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Optimal exploitation of a renewable resource with capital limitations: Nordic sheep farming with and without grazing externalities

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Abstract

A model of interaction between a renewable natural resource with capital limitations, as exemplified by the optimal investment problem of sheep farming in a Nordic context, is analysed. Both private and social optimality are considered; with the difference that a stock value related to the number of grazing animals is attached to the social management problem due to landscape preservation. The efficiency of alternative policy tools in terms of obtaining the socially optimal management scheme is discussed. The model builds on existing studies from the fisheries literature, but the important difference is that while capital is related to harvesting effort in the fisheries, capital contributes to production capacity to keep the animal stock during the winter in our farm model. The paper provides several results where both optimal steady states and the optimal approach paths are characterised analytically. The results are further supported by a numerical example.

Keywords: livestock management, irreversible investment, optimal control, grazing externalities

1. Introduction

Following the pioneering work of Smith (1968), economic models of renewable resource management have occasionally been extended to include investment in man-made capital. Even though most, if not all, contributions to this strand of literature have been related to fishery management problems, spurred by the seminal contribution of Clark, Clarke and Munro (1979), much of the conclusions obtained here can probably quite easily be extended into the management of other types of wild natural resources, like terrestrial wildlife. In this paper, we look at another type of renewable management problem with capital limitations, namely domestic livestock management. The important difference compared with fisheries is that while capital in the fishery is related to the capacity of

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harvesting animals, it is related to *keeping* animals in our livestock farming problem, which is exemplified by sheep farming in a Nordic context.

The literature on the management of what may be viewed as two capital stocks, one man-made and the other one biological, is quite small. In most cases, man-made capital is implicitly assumed malleable enough to be treated as a variable cost. However, the issue in our model is that investments will be sunk to a large extent; it is hard to lease buildings and related equipment once constructed. In the fishery model, Clark *et al.* (1979) emphasise this irreversibility of investment, meaning that man-made capital cannot be sold once having been bought, and they show how the possible approach paths towards the optimal steady state is greatly affected by this property. Their model is linear in both controls, investment in fishing vessels and harvest of the fish stock, and the approach paths are therefore characterised by a combination of bang–bang and singular controls. Stochastic elements are included in a paper by Charles and Munro (1985), and McKelvey (1985) analyses open access dynamics in a fishery with man-made capital. Boyce (1995) formulates a similar model to that of Clark *et al.* (1979), but with non-linear investment costs. He finds, not surprisingly, that the derived optimal approach path is no longer of the bang–bang type. Sandal *et al.* (2007) extend the literature with a model without any non-negative constraint on investment, but where capital is less valuable when sold than when bought.

In this paper we analyse the optimal investment and harvest, or stocking, decision problem of a sheep farmer. The farmer, assumed to be well-informed and rational, aims to maximise present-value profit generated by meat production. The market price of meat is taken as given, as we consider a single farm, and abstract from both exogenous price fluctuations and (other) stochastic factors such as climatic variations. In addition to the natural capital stock, the animals, the farmer must also hold a certain amount of man-made capital which adheres to the familiar mechanisms of investment and depreciation, to keep the animals indoor during the winter season because of harsh weather conditions. Man-made capital in this farming system is thus mainly buildings and related equipment which is instrumental in determining farm capacity. Once invested, this is sunk cost as it is hard to lease. In addition to the costs and benefits of the individual farmer, we also analyse the situation where a positive value is attached to the sheep stock. This reflects the amenity value of the cultural landscape, which is preserved by grazing. Both in Norway and in the EU, the amenity value of the landscape is an argument for agricultural support ('the multifunctional value of agriculture', see, e.g. Brunstad, Gaasland and Vaardal, 2005), and we look at how different policy instruments, e.g. stock subsidy and meat price (slaughter) subsidy, may increase the optimal animal stock.

We are not aware of other theoretical domestic livestock management models that include man-made capital in addition to animal capital, even though capital theoretical treatments of livestock are frequently found within the resource economics literature, see, e.g. Kennedy (1986). Farm models include Jarvis (1974) who formulated a timing problem of cattle grazing, and Skonhoft (2008) who analysed the optimal stocking problem of Nordic sheep farming. Our model and reasoning builds to some extent on this last paper, but Skonhoft studied a situation with no man-made capital limitations and with different year classes of the animal capital. Different year classes are not included in the present paper. The research problem here is to find the optimal slaughtering and investment policy in such a Nordic farming system, and to characterise both the optimal steady state and approach paths. In the subsequent analysis, natural and man-made capital will generally be referred to merely as 'animals' and 'capital', respectively.

The paper is organised as follows. Section 2 describes briefly the Nordic sheep farming system and the model is formulated. Section 3 analyses the optimal solution to the individual farmer problem where both the steady states and the dynamics are considered. In Section 4 the social planner model is studied and possible policy instruments are analysed. Numerical simulations are shown in Section 5, while Section 6 concludes the paper. The details of the dynamic analysis are given in the Appendix where we show that the optimal dynamic management policy involves a combination of bang–bang and singular controls.

2. Model

The following analysis is related to economic and ecological conditions found in Norway, but these also exist in Iceland, Greenland and other places with snow and harsh winter conditions. There are about 2.1 million sheep in Norway during the summer grazing season. More than 1 million of these are slaughtered in the fall. Nearly all male lambs are slaughtered because of the controlled breeding system, so the winter stock counts somewhat >1 million mainly female animals (ewes). There are about 14,000 farms, all family farms, and the average farm size is therefore quite small and accounts only for about 150 summer grazing animals. Sheep farms are located either close to mountain areas and other sparsely populated areas or along the coast, with a means to transport the animals to more distant alpine areas with access to areas of summer grazing land. Such land is typically communally owned and managed. There is a sharp distinction between the summer grazing season and the winter indoor season. While food is abundant during the summer grazing season, housing and indoor feeding is required throughout winter. The indoor winter season is typically from mid-October to the beginning of May next year. The adult sheep and the newborn lambs are then released for outdoor grazing. In September–October, slaughtering takes place. In Norway, winter feeding basically consists of hay grown on the farms, with the addition of concentrate pellets provided by the industry. The main product is meat, which accounts for about 80 per cent of the average farmer's market income. The remainder comes from wool, because sheep milk production is virtually non-existent (Nersten *et al.*, 2003). However, income from wool is neglected in the following analysis. Norwegian sheep farming is also heavily subsidised, through a complex package of subsidies (Nersten *et al.*, 2003).

We begin with formulating the animal growth equation, given in discrete time with a time resolution of 1 year, and with a seasonal subdivision between the outdoor grazing period (spring, summer and fall) and indoor winter feeding period. The population is measured in the winter and in the beginning of the year. In the single period of 1 year, three events occur in the following order: first lambing, then natural mortality, and finally slaughtering. Therefore, when neglecting the small natural mortality during the indoor season, all mortality is assumed to take place during the outdoor grazing season related to various diseases, accidents and predation by large carnivores. As our main focus is on the interaction between biological and man-made capital, we do not distinguish between different age classes of animals (see Skonhoft, 2008), but consider a biomass model where 'a sheep is a sheep'. The rate of growth in animal biomass is further assumed constant, as is reasonable with a domestic animal stock facing controlled breeding and maintenance; that is, there is no density-dependent growth process. The growth function for animal biomass may thus be written as follows:

$$X_{t+1} - X_t = rX_t - H_t, \tag{1}$$

such that X_t is the animal stock size in the beginning of year t, $H_t \geq 0$ is harvest, taking place after natural growth and $r > 0$ is the animal stock net natural growth rate, assumed to be constant. The size of the winter stock is thus X_t while the summer stock at the end of the grazing season, but before slaughtering, is $(1 + r)X_t$.

Man-made capital, also assumed to be homogenous, is used as housing for the animal stock during the winter indoor season. Each year, a limited positive amount of investment is allowed, and a constant fraction of the capital stock depreciates due to wear and tear. The net capital growth is thus given by

$$K_{t+1} - K_t = -\gamma K_t + I_t, \tag{2}$$

where K_t is the capital stock and I_t is the accompanying (gross) investment. $\gamma > 0$ is the rate of depreciation, assumed to be fixed. This formulation implies, among others, that in a steady state with a fixed amount of capital over time, investment will equalise depreciation every year. In real-life situations, however, we would not expect that a 'small' investment would take place every year, but be relatively 'large' in some years and possibly zero in most years. However, in our model, as in Clark *et al.* (1979), the fact that investments possibly would be more lumpy in nature is not taken into account.

The revenue of the farmer is made up of income from meat production. With $p > 0$ as the slaughtering price (net of slaughtering costs), the current meat income for the farmer simply reads pH_t and is included as the first term in the profit equation:

$$\Pi_t = pH_t - V(X_t) - Q(X_t, K_t) - cI_t. \tag{3}$$

p is assumed fixed over time and independent of the harvest decision, as explained above (Section 1).

The cost side is made up of operating cost, $V(X_t)$, congestion cost, $Q(X_t, K_t)$ and investment cost, cI_t. The operating cost structure differs sharply between the outdoor grazing season and the indoor feeding season. As explained, during the grazing period the sheep graze on communally or private-owned land in the valleys and the mountains. The animal density is generally low and possible overgrazing problems may accrue only to some few areas (Austrheim *et al.*, 2008) and are hence not considered as a problem here. Such land may be available cost free, or the farmer may pay a fixed annual rent (Austrheim *et al.*, 2008). For this reason, we do not include an explicit land size constraint, and the variable cost is simply assumed to be the indoor season-operating cost. These costs, which include labour cost (typically as an opportunity cost), electricity and veterinary costs in addition to fodder, are assumed to be determined uniquely by the size of the winter animal stock, i.e. $V_t = V(X_t)$, and with $V'' \geq 0$, $V'' \geq 0$, and $V(0) = 0$. The traditional argument for a strictly convex cost function is that fodder production is, at least in the short run, constrained by the size of the available land; that is, as the stock becomes larger it may become progressively more costly to provide winter fodder grown on pastures close to the farm. Alternatively, linear-operating cost reflects the situation where additional fodder, as well as other inputs, can always be bought at a constant market price.

As mentioned, in contrast to what is found in the fisheries literature where capital is an input into harvesting effort (e.g. Clark *et al.*, 1979), capital in our farm system is housing and related equipment to keep the animal stock during the winter. We assume that there is no absolute constraint on the amount of animals that a given amount of capital can support, so that there is no such thing as 'full' capacity utilisation in our farm model. However, as the indoor space per animal diminishes, the operating procedure becomes increasingly cumbersome. We hence include a capacity utilisation cost, or congestion cost, function $Q(X_t, K_t)$ in our current profit equation (3). It increases with the number of the winter stock, for any given amount of capital, such that $Q_x > 0$ and $Q_{xx} > 0$, together with $Q_K < 0$, $Q_{KK} > 0$ and $Q_{KX} < 0$. In addition, we have $Q(0, K_t) = 0$ when $K_t > 0$ and $\lim_{K_t \to 0} Q(X_t, K_t) = \infty$ for $X_t > 0$. For all positive stock values, this function is hence convex in X_t and K_t. In Section 5, we specify this cost function.

The final private cost component is investment in new capital equipment. We assume, in contrast to Clark *et al.* (1979), that there is a constraint on the size of investment in each period, due to, say, limited access to credit, debt aversion, lack of capacity for providing construction services locally and so forth. We therefore have $I_t \leq I^{\max}$, where $I^{\max}$ is assumed to be constant. A more sophisticated approach could be to let the maximum investment depend positively on the already existing capital, stock used as collateral for new loans. Yet another alternative, following, e.g. Sandal *et al.* (2007), could be to introduce adjustment costs to limit the amount of investment carried out in each time period. In our model, as in reality, investment is also irreversible ('nonmalleability'); the buildings have few, if any, alternative uses once having been set up; that is, $I_t \geq 0$. The cost per unit of investment is fixed and given by $c > 0$, so that the yearly investment cost reads cI_t.

As mentioned, we also look at the situation where, from society's point of view, a positive value is attached to the sheep stock of the individual farmer. This reflects the amenity value of the landscape and that grazing keeps the landscape open and avoids shrubs and weeds becoming established. In the model, this positive externality is introduced by adding the positive stock value $W_t = \tilde{W}((1+r)X_t)$, and where the summer grazing stock is measured at the end of the grazing season (see above). More conveniently we write this as follows:

$$W_t = W(X_t), \tag{4}$$

with $W' > 0$, $W'' \leq 0$ and $W(0) = 0$, and where the fixed natural growth factor $(1 + r)$ is embodied in the functional form W. The net social benefit produced by the considered individual farmer is thus equation (3) plus the positive stock value (4).

3. Optimal private management

3.1 The problem of the farmer

The farmer aims to maximise present-value profit subject to the dynamic constraints imposed by the growth equations for animals (1) and capital (2), and the constraints on harvest and investment in each period. We suppose an infinite planning horizon, meaning that we are looking for an optimal steady state. The planning problem of the farmer is then formulated as follows:

$$\max\left\{\sum_{t=0}^{\infty}\rho^t[pH_t - V(X_t) - Q(X_t, K_t) - cI_t]\right\}$$

$$\begin{aligned} s.t.\quad & X_{t+1} - X_t = rX_t - H_t \\ & K_{t+1} - K_t = -\gamma K_t + I_t \\ & 0 \leq H_t,\ \ 0 \leq I_t \leq I^{\max} \\ & X_t, K_t > 0 \\ & X_0, K_0 \text{ given,} \end{aligned} \tag{5}$$

and where $\rho = 1/(1+\delta)$ is the discount factor with $\delta \geq 0$ as the constant discount rate.

The Lagrangean of this problem may be written as follows:

$$\begin{aligned} L = \sum_{t=0}^{\infty}\rho^t\ \{ & pH_t - cI_t - V(X_t) - Q(X_t, K_t) \\ & - \rho\lambda_{t+1}[X_{t+1} - (1+r)X_t + H_t] \\ & - \rho\mu_{t+1}[K_{t+1} - (1-\gamma)K_t - I_t]\}, \end{aligned}$$

where λ_t and μ_t are the shadow prices of the animal and capital stock, respectively. The necessary conditions for a maximum are as follows:

$$\frac{\partial L}{\partial H_t} = p - \rho\lambda_{t+1} \leq 0, \quad 0 \leq H_t, \tag{6}$$

$$\frac{\partial L}{\partial I_t} = -c + \rho\mu_{t+1} \underset{\leq}{\overset{>}{=}} 0, \quad 0 \leq I_t \leq I^{\max}, \tag{7}$$

$$\frac{\partial L}{\partial X_t} = -V' - Q_X + \rho\lambda_{t+1}(1+r) - \lambda_t = 0 \tag{8}$$

and

$$\frac{\partial L}{\partial K_t} = -Q_K + \rho\mu_{t+1}(1-\gamma) - \mu_t = 0. \tag{9}$$

These conditions are also sufficient if the Lagrangean is concave in the states and controls jointly. Since the Lagrangean is linear in the controls, the sufficiency conditions boil down to $L_{XX} = -(V'' + Q_{XX}) \leq 0, \quad L_{KK} = -Q_{KK} \leq 0$, which are always satisfied for the given properties of the cost function and $L_{XX}L_{QQ} - {L_{XY}}^2 = Q_{KK}(V'' + Q_{XX}) - {Q_{KX}}^2 \geq 0$, which is satisfied at the optimal steady state (see discussion below). The transversality conditions for the infinite horizon problem must also hold; i.e. $\lim_{t\to\infty} \rho^t \lambda_t X_t = 0$ and $\lim_{t\to\infty} \rho^t \mu_t K_t = 0$.

Relationships that define the interior, or 'singular' controls for both stocks are derived from the first-order conditions as follows. If singular harvest holds, we have from equation (6) $p = \rho\lambda_{t+1}$, which means that the shadow price of the animal stock is constant and equalises $\lambda = p/\rho$. When this expression is inserted into equation (8), we find the following golden rule condition for the animal stock:

$$(r - \delta)p = V'(X_t) + Q_X(X_t, K_t). \tag{10}$$

Equation (10) therefore describes the relationship between X_t and K_t that is consistent with singular harvest. This condition may also be written as $p = (1/\delta)[pr - V'(X_t) - Q_X(X_t, K_t)]$, indicating that the market revenue from selling one animal should equalise the discounted net benefit from keeping it. Because both $V'$and Q_X are positive, we must require that the animal growth rate exceeds the discount rate, $r > \delta$, which is a well-known condition for a positive steady-state animal, or fish, stock (see, e.g. Clark, 1990). As both r and δ are constant, this must always hold, also outside the steady state.

With singular investment, we have from (7) $c = \rho\mu_{t+1}$, which means that the capital shadow price is constant, $\mu = c/\rho$. Inserted into equation (9), this gives the optimality condition for capital

$$(\gamma + \delta)c = -Q_K(X_t, K_t). \tag{11}$$

Equation (11) thus defines a relationship between X_t and K_t, that is consistent with singular investment. It may also be written as $c = (1/\delta)[-Q_K(X_t, K_t) - \gamma]$, indicating that the unit investment cost should equalise the discounted marginal net benefit from holding capital.

Equation (10) gives X_t^S, the animal stock consistent with singular harvest, as determined by K_t, and describes a curve in the (K, X) space along which singular harvest holds. Using equation (1) then obtains the singular harvest rule that moves the system along this path towards the equilibrium, as a function of the two stocks, $H^S(X_t, K_t)$, see also Section 3.3. Starting off this path, harvest is set either at zero or maximum until the system reaches this singular trajectory. We thus have the following alternatives for harvest policy:

$$H_t = \begin{cases} (1+r)X_t - X_t^S & \text{when} \quad p > \rho\lambda_{t+1} \\ H^S & \text{when} \quad p = \rho\lambda_{t+1}. \\ 0 & \text{when} \quad p < \rho\lambda_{t+1} \end{cases}$$

Note that, since there is no upper bound on harvest except from the size of the stock itself, whenever $p > \rho\lambda_{t+1}$, the herd will be reduced immediately (that is, within one-time period), to X^S, where $p = \rho\lambda_{t+1}$, and singular harvest takes over. This is often called impulse control in the optimal control theory literature.

Similarly, condition (7) states that investment is set to its upper or lower boundary in order to move the capital stock onto a singular investment trajectory where the unit investment cost equals the discounted capital stock shadow price

$$I_t = \begin{cases} \min\{I^{\max}, K_t^S - (1-\gamma)K_t\} & \text{when} \quad c < \rho\mu_{t+1} \\ I^S & \text{when} \quad c = \rho\mu_{t+1}. \\ 0 & \text{when} \quad c > \rho\mu_{t+1} \end{cases}$$

Therefore, investment will be at its maximum, provided that this is less than the difference between the existing capital stock and the one corresponding to singular investment, or minimum level whenever the per unit investment cost is lower or higher than the discounted shadow price of capital. When the shadow price reaches the point where it equals the present value of the unit cost of investment, the control will either switch between the two control boundaries – going from maximum to zero investment, or vice versa – or stay at singular investment for some amount of time.

It is interesting to compare the optimal control regimes in the present model with the standard one-state variable framework. In a one-state linear control problem, the singular control would correspond to a single point, which is the optimal interior steady state. However, singular policies are defined along curves in the K, X space in our model. These curves act as possible approach paths to the optimal steady state, where the interior control corresponding to the chosen path is singular also outside the steady state. Furthermore, given the linearity of the model, the singular approach path is optimally reached as fast as possible; a generalisation of the most rapid approach path solution normally found in linear control problems.

3.2 The steady states

In an interior equilibrium where both $H^* = H^S(K^*, X^*)$ and $I^* = I^S(K^*, X^*)$, where superscript '*' indicates optimal steady-state values, the golden rule equations (10) and (11) must hold. But in principle, one, or both, controls may also be set at a boundary at a steady state. From equation (1), however, as long as the rate of animal growth is positive and constant, the harvest rate must be positive in the steady state. Since there is no upper constraint on harvest, the steady-state harvest policy must then be singular. From equation (2), steady-state investment must also be positive, but may be set to its maximum level, where the gross investment in each year equals depreciation, keeping the capital stock at its optimal steady-state level given the investment constraint. We therefore have two alternatives for the steady state, and this is stated as follows:

Result 1. There are two steady-state alternatives. The first is interior where both harvest and investment are singular. In the second steady-state harvest is singular while investment is at the maximum level.

We first study the interior steady state in some detail and then discuss the situation where the investment constraint binds. At an interior steady state, the two schedules defined by equations (10) and (11) must intersect. Except from the very special case where the two curves are coinciding, there can be at most a countable number of equilibria. When differentiating (10) and (11), we find $\mathrm{d}X/\mathrm{d}K = -Q_{XK}/(V'' + Q_{XX}) > 0$ and $\mathrm{d}X/\mathrm{d}K = -Q_{KK}/Q_{XK} > 0$, respectively. Therefore, both schedules (10) and (11) slope upwards in the (K, X) space, but the curvatures cannot be determined generally without imposing restrictions on third derivatives. This allows for an arbitrary number of intersection points, with a correspondingly arbitrary number of stable and unstable equilibria. However, we find a locally optimal steady-state combination of animals and capital, denoted K^* and X^*, where the singular harvest condition (10) intersects the singular investment schedule (11) from above, so that $-Q_{XK}/(V'' + Q_{XX}) > -Q_{KK}/Q_{XK}$ holds at the intersection point. See Figure 1 where equations (10) and (11) are based on equations (10′) and (11′) (Section 5). Otherwise, the intersection point is an unstable equilibrium. This holds because a local maximum is found where the Lagrangean is concave around a stationary point, which requires that the condition $(V'' + Q_{XX})Q_{KK} - {Q_{KX}}^2 = \Gamma(X, K) \geq 0$ must be satisfied. Rearranging this expression gives $-Q_{XK}/(V_{XX} + Q_{XX}) > -Q_{KK}/Q_{XK}$ as claimed. When these equilibria are found, the steady-state harvest follows from equation (1) as:

$$H^* = rX^* \tag{12}$$

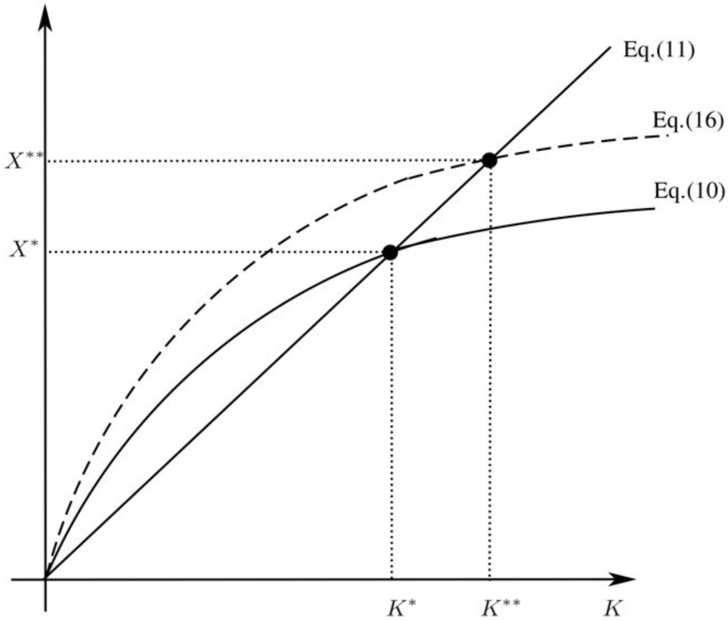

Fig. 1. Interior steady states. Private optimisation (K^*, X^*) and social planner solution (K^{**}, X^{**}).

and the steady-state investment from equation (2) as:

$$I^* = \gamma K^*. \tag{13}$$

Equations (10) and (11) and the sufficiency condition can be used to derive some comparative static results about X^* and K^*. In a next step, the effects on I^* and I^* follow recursively from equations (12) and (13), respectively. We first look at the effect of a changing meat price, and when differentiating equations (10) and (11) we find $(r - \delta)\,\mathrm{d}p = (V'' + Q_{XX})\mathrm{d}X + Q_{KX}\mathrm{d}K$ and $0 = -Q_{KK}\mathrm{d}K - Q_{KX}\mathrm{d}X$, respectively. Combining these expressions yields the partial price effects $\partial X^*/\partial p = Q_{KK}(r - \delta)/\Gamma(X, K) > 0$ and $\partial K^*/\partial p = -Q_{KX}(r - \delta)/\Gamma(X, K) > 0$. This is stated as follows:

Result 2. An increase in the price of meat will result in a larger stock of animals and capital in an interior optimal steady state.

This result is the opposite of what is found in the standard fishery model (e.g. Clark, 1990) where a price increase leads to more aggressive harvest and a lower optimal steady-state stock. The main reason for the opposite result in our farm model is that costs here are not associated with harvest, but with stock maintenance. With a higher meat price, the farmer thus finds it beneficial to keep a higher stock of both capital and animals as the maintenance cost decreases relative to the meat price. A positive shift in equation (10) occurs, similar to what is depicted in Figure 1. Differentiating equations (10) and (11) also gives

Table 1. Comparative static results interior steady state with singular harvest and investment

	p	c	δ	r	γ
X^*	$+\ (+)$	$-\ (0)$	$-\ (-)$	$+\ (+)$	$-\ (-)$
K^*	$+\ (0)$	$-\ (0)$	$-\ (0)$	$+\ (0)$	$-\ (-)$
H^*	$+\ (+)$	$-\ (0)$	$-\ (-)$	$+\ (+)$	$-\ (-)$
I^*	$+\ (0)$	$-\ (0)$	$-\ (0)$	$+\ (0)$	$\pm\ (0)$

The constrained steady state with $I^* = I^{\max}$ in brackets.

information about the effects of a change in the discount rate. We find positive effects for both stocks, and this is stated as follows:

Result 3. An increase in the discount rate leads to reduced stocks of both animals and capital in the interior optimal steady state.

This result fits conventional economic intuition, but is far from obvious when more than one capital stock is included. As shown by, e.g. Asheim (2008), paradoxical effects of discounting, such as a positive relationship between discounting and steady-state consumption, may result from multi-dimensional models. Also, when there is a trade-off between the two stocks across alternative steady states, something that is typical for predator–prey models, one of the stocks must increase with discounting while the other goes down. However, this does not happen here.

Following the same procedure with respect to the other parameters, all the time assuming that the sufficiency conditions are fulfilled, the other comparative static results can also be computed. All the results are reported in Table 1 where the investment and harvest effects are included as well. An increase in the investments cost or depreciation rate means that it is beneficial for the farmer to reduce the steady-state animal and capital stocks, whereas an increase in the growth rate of animals leads to larger optimal stocks of both animals and capital. These results are more or less as expected, and the effects of the parameters work in the same direction for both stocks. It can also be confirmed that we find the combined effects $\partial X^*/\partial r = \partial X^*/d\gamma - \partial X^*/\partial\delta > 0$, indicating that the negative effect on the animal stock of a higher depreciation rate must be smaller than the negative effect of a higher discount rate. We also find $\partial K^*/\partial r = \partial K^*/d\gamma - \partial K^*/\partial\delta > 0$, indicating that the same holds with respect to the optimal steady-state capital stock.

The effects on the control variables through equations (12) and (13) are quite straightforward, except for the animal growth rate and rate of depreciation which both have direct and indirect effects on the steady-state harvest and investment, respectively. For a positive shift in the animal growth rate, the two effects work in the same direction and lead to more harvest in the steady state, since from equation (12) $\partial H^*/\partial r = X^* + r\partial X^*/\partial r > 0$. With the depreciation rate, however, the two effects work in the opposite direction as we find $\partial I^*/\partial\gamma = K^* + \gamma\partial K^*/\partial\gamma$ from equation (13). The direct effect is to increase the required amount of investment to maintain a given amount of capital, whereas the indirect effect is to decrease the optimal steady-state capital stock. The overall effect is ambiguous with general functional forms.

We then consider the other steady-state possibility where the investment is no longer singular, and hence the condition

$$I^* = I^{\max} < I^S \tag{14}$$

replaces equation (13) in the interior steady-state solution. The steady-state capital stock now follows directly through (13) as $K^* = I^{\max}/\gamma$, which inserted into equation (10) yields the steady-state animal stock. The amount of capital will now for obvious reasons be below what was found in the interior steady state. Because schedule (11) yields a positive relationship between the two stocks, the number of animals will also be below what was found in the interior steady state. The steady-state harvest again follows from equation (12) as $H^* = rX^*$, and the number of animals slaughtered will consequently also be below the previous steady state.

The comparative static results are now somewhat different, as indicated with brackets in Table 1. When the investment constraint binds, the depreciation rate is the only factor that affects the steady-state capital stock. The signs of the effects on the animal stock are as before, except for the investment cost which now has zero effect.

3.3. Optimal approach paths

We have characterised the two alternatives for an optimal steady state, both the interior solution and the case where the upper investment constraint binds at the optimum. The next task is to study the optimal approach paths. In general, approach paths in multi-dimensional models are often complicated to analyse, as exemplified by the predator–prey model of Mesterton-Gibbons (1996). For a more recent example, see the infectious wildlife disease model in Horan and Wolf (2005). We find, however, that in our case it is possible to derive an intuitive solution which is explained graphically in Figure A1 in the Appendix.

By differentiating equations (10) and (11), and using the growth equations to substitute for $K_{t+1} - K_t$ and $X_{t+1} - X_t$, we can derive explicit dynamic feedback rules for both stocks. As there is no upper constraint on the harvest, so that $H_t = 0$ along the singular investment schedule, singular investment is given by

$$I_t^S = \gamma K_t - \frac{Q_{XK}}{Q_{KK}} r X_t,$$

with $Q_{XK}/Q_{KK} < 0$. Singular investment therefore depends positively on depreciation, and also on animal stock growth. A higher stock of animals, and/or a higher animal growth rate means that investment must increase to let capital growth keep pace with growth in the animal stock. Singular harvest is given by

$$H_t^S = \begin{cases} rX_t - \dfrac{Q_{XK}}{V'' + Q_{XX}}\gamma K_t & X_t > X^* \\ rX_t - \dfrac{Q_{XK}}{V'' + Q_{XX}}(\gamma K_t - I^{\max}) & X_t < X^* \end{cases},$$

where $Q_{XK}/(V'' + Q_{KK}) < 0$, and with a similar interpretation. Note that the singular harvest rule is different depending on whether investment is set to its upper or lower boundary.

Three results regarding the optimal dynamic adjustment process are stated here without proof (but see the Appendix). The first two results are related to the monotonicity of the approach paths, and can be compared with the results from the fisheries literature. The first of these results is stated as follows:

Result 4. It is never optimal for capital to overshoot its optimal steady-state level. However, capital may undershoot the steady state if $X_0 < X^*$.

Corollary: It will never be optimal to have excess capacity in the steady state.

This result differs from what is found by Clark *et al.* (1979), where it is optimal to have excess capacity in the steady state if the depreciation rate is zero. The reason that this does not happen here is that capital plays no role in the harvesting process. Therefore, it is not profitable, or even possible, to speed up the approach to the equilibrium by overinvesting, if the initial animal stock is above the equilibrium level. The next result concerns the development of the animal stock and is stated as follows:

Result 5. The animal stock may either undershoot or overshoot the optimum, depending on the initial situation.

This also contrasts the Clark *et al.* (1979) model, and subsequent contributions within the fishery economics literature. The intuition is that a more profitable rate of capacity utilisation can be obtained by temporarily reducing the animal stock below the steady-state level if the capital stock is low, and expanding it beyond the steady-state level if the capital stock is large. Both situations depend on the fact that the capital stock cannot be adjusted instantaneously in either direction. A last result from the dynamic analysis is stated as follows:

Result 6. If the upper investment constraint is not binding on the approach path, the optimal steady state will generally be approached with one control set at the interior and the other at zero.

In principle, all control combinations are possible approaches to the equilibrium, but the case where $X_0 > X^*$ and $K_0 = K^*$, so that the equilibrium is reached by a one-time slaughtering down of the animal stock only, and the case where the equilibrium is reached by setting both controls to zero, can both only be satisfied by a fluke. The general approach is along one of the singular control schedules, and ruling out the possibility that $I = I^{\max}$ along the approach path, the non-singular of the controls must be zero.

4. Optimal management: social planner

Assume now that the animal stock of the individual farmer represents a public good value as given by equation (4) because of landscape preservation. In reality, farms are heterogeneous and some typically contribute more to the preservation than others. In this model, however, we consider a 'representative', or average farmer, where the contribution to landscape preservation depends only on the number of animals, and not on location, etc. We then simply state the social benefit as the sum of the private profit of the farmer and the public good value of the animal stock. Therefore, the social planner maximises the social surplus

$$U_t = pH_t - V(X_t) - Q(X_t, K_t) - cI_t + W(X_t) \tag{15}$$

discounted over an infinite time horizon as before. Repeating the procedure followed in Section 3.1, the equivalent of the golden rule equation (10), when the social planner has the same discount rate as the farmer and in a steady state where time subscripts can be dropped, now reads

$$(r - \delta)p = V'(X) + Q_X(X, K) - W'(X), \tag{16}$$

while equation (11) still prevails.

It can easily be demonstrated that inclusion of the external animal stock value shifts equation (16) up compared with equation (10) in the K,X plane. We only consider the interior stable steady state where the new optimality condition (16) intersects with equation (11) from above. Not surprisingly, we then find that it is optimal with more animals and also more capital in the social planner solution than in the private optimisation problem; that is, $X^{**} > X^*$, and $K^{**} > K^*$, where superscript '**' now denotes the interior steady-state social planner solution. Thus, taking the positive animal stock externality into account means that the social planner solution not only demands more animals, but also more capital. See Figure 1 where equation (16) is based on (16′) in Section 5.

Given that the socially optimal stocks of both animals and capital differ from what is desired by the privately optimising farmer, there are two separate policy targets to be achieved by the optimal policy. In general, this would require two separate policy instruments. However, because only the animal stock has a social value that directly exceeds that of the private owner, a first best social optimum may be obtained by just targeting the animal stock by imposing a subsidy per living animal. With the animal stock subsidy $s^X > 0$ (EUR/animal), the current profit function of the farmer reads $\Pi_t(s^X) = pH_t + s^X X_t - cI_t - V(X_t) - Q(X_t, K_t)$. The equivalent of the optimality equation (10) then becomes $(r - \delta)p + s^X = V'(X) + Q_X(X, K)$. When comparing with the social planner solution equation (16), we hence find that the optimal animal stock subsidy in the steady state should equalise $s^X = W'(X^{**})$. The socially optimal capital stock is then automatically obtained as there are no externalities pertaining directly to the capital stock; that is, equation (11) is left unchanged. The marginal per animal stock subsidy is given as a fixed marginal stock value here, but to guarantee that the socially optimal solution is followed also along the transition path a more flexible subsidy scheme would have to be employed, so that s^X is allowed to vary with X_t to always equalise $W'(X_t)$. In reality, however, a constant per animal subsidy is probably the most viable option. The same caveat applies to the other subsidy schemes considered below.

Another policy option may be to introduce a meat price (slaughter) subsidy, $s^p > 0$ (EUR/animal). The current profit of the farmer in this case reads $\Pi_t(s^p) = (p + s^p)H_t - cI_t - V(X_t) - Q(X_t, K_t)$. The equivalent of the optimality condition (10) now becomes $(r - \delta)(p + s^p) = V'(X) + Q_X(X, K)$ while condition (11) still is left unchanged. Therefore, also in this case it is possible to obtain two policy goals with only one instrument. In light of condition (16), we find that the optimal steady-state meat price subsidy should equalise $s^p = W'(X^{**})/(r - \delta)$.

It is also of interest to compare the efficiency of these two policy instruments (for a related discussion see, e.g. Schulz and Skonhoft, 1996). With the price subsidy, the total yearly steady-state subsidy is $W'(X^{**})/(r - \delta)H^{**} = W'(X^{**})/(r - \delta)\, rX^{**}$, while in the stock subsidy case becomes $W'(X^{**})X^{**}$. With $\delta > 0$, and still $(r - \delta) > 0$, we thus find that the steady-state price subsidy exceeds that of the stock subsidy. This is stated as follows:

Result 7. Evaluated at the interior steady state, the stock subsidy is more efficient than the meat price subsidy as the identical optimal allocation can be achieved with lower costs for the regulator.

In terms of Figure 1 [based on equations (10′), (11′) and (16′)], the social optimum lies along the singular investment schedule, and is thus obtained by shifting the singular harvest schedule only by using the stock subsidy. This can also be done by way of the meat price subsidy, but in general at a higher cost for the regulator, as demonstrated above. As shown in Section 5, however, the cost disadvantage of the price subsidy seems to be quite modest, as the difference between r and δ is substantial in the present farm management system. Therefore, a meat price subsidy may serve as a reasonably good second best policy option, in particular if it is more easily accepted and implemented.

A physical investment subsidy $s^c > 0$ (EUR/m^2) is also a possible policy option. The current profit function of the farmer then reads $\Pi_t(s^c) = pH_t - (c - s^c)I_t - V(X_t) - Q(X_t, K_t)$. In this situation, the optimality condition (10) is unchanged while the equivalent of equation (11) now becomes $(\gamma + \delta)(c - s^c) = -Q_K(X, K)$. The working of this investment subsidy, however, differs from the above considered subsidy schemes as it is now not possible to reach both policy goals X^{**} and K^{**} with just one instrument. The reason is that while the externality is related to the animal stock, the capital subsidy is related to the capital stock. Again in terms of Figure 1, we find that while the animal stock externality shifts up equation (10), introduction of s^c shifts down equation (11). Thus, by introducing s^c the socially optimal steady-state capital stock can be obtained, or the social optimal steady-state animal stock can be obtained, but not both goals at the same time.

5. Numerical example

5.1 Functional forms and data

To shed some further light on the above analysis, the model is now illustrated numerically.[1] We do not attempt to accurately describe the economic situation of a Nordic sheep farmer, but to demonstrate the workings of the model with reasonably realistic parameter values. First, we specify the functional forms. The congestion cost function is specified as $Q(X_t, K_t) = (\theta/2K_t)X_t^2$, where $\theta > 0$. It is readily confirmed that this cost function satisfies the properties stated in Section 2. The operating cost function is next specified strictly convex as $V(X_t) = (\eta/2){X_t}^2$, with $\eta > 0$, while the stock externality function (4) is specified linear, $W_t = wX_t$, with $w > 0$, and where $(1 + r)$ is embedded in w as the stock externality is measured in the summer grazing season, i.e. $w = \tilde{w}(1 + r)$.

With these functional forms, we find the following expression for the singular harvest and investment schedules in the private optimisation problem:

$$(r - \delta)p = \left(\eta + \frac{\theta}{K_t}\right)X_t \tag{10'}$$

and

$$(\gamma + \delta)c = \frac{\theta}{2}\left(\frac{X_t}{K_t}\right)^2. \tag{11'}$$

It is easily recognised that both schedules start from the origin and have positive slopes. While the singular investment schedule (11′) is a straight line, the singular harvest schedule (10′) yields X as a strictly concave function of K, cf. Figure 1. They have thus one interior intersection point, provided that the H^S-schedule is steeper than the I^S-schedule at the origin which holds for the given parameter values, and this corresponds to a stable equilibrium (see also Section 3.2). In the social planner problem, equation (16) reads as follows:

$$(r - \delta)p = \left(\eta + \frac{\theta}{K_t}\right)X_t - w. \tag{16'}$$

The numerical optimisation is performed using the parameter values found in Table 2. The discount rate and the indoor-feeding cost parameter are taken from Skonhoft (2008), while the depreciation rate is what is used by Statistics Norway for buildings (Statistics Norway, 2011). As for the assumed growth rate of the animal stock and the slaughter (meat) price, an explanation is in order. With on average about 1.5 lambs per ewe and modest natural mortality

1 The numerical optimisation was performed using the KNITRO for MATLAB solver form Ziena Optimization, with MATLAB release 2011b.

Table 2. Baseline parameter values

Parameter	Description	Value
δ	Discount rate	0.04
r	Animal growth rate	0.7
c	Unit investment cost (EUR/m^2)	100
η	Feeding cost (EUR/animal^2)	1.1
θ	Congestion cost (EUR/(animal^2/m^2))	45
γ	Depreciation rate	0.04
p	Meat price (EUR/animal)	240
w	Stock externality (EUR/animal)	20

Sources and assumptions; see main text.

(see Skonhoft, 2008), we suppose that the number of animals that can be slaughtered in the fall each year in a steady state is 1.4 times the winter stock. But as the winter stock consists of ewes only (Section 2), it can only grow at a maximum rate of 0.7 when half of the lambs are females. We solve this by setting $r = 0.7$ and doubling the meat price compared with the actual price (obtained from Skonhoft, 2008) to get a realistic picture of the annual slaughtering profit. For our purpose this suffices, but for more realistic results, for instance with respect to profits outside the steady state, a more elaborate modelling of the controlled breeding system would be required including different age and sex categories of the animals. The investment and congestion cost parameters are calibrated for our model such that the number of animals in the steady state should represent a rather large sized Norwegian farm. In addition, we assume that the maximum yearly investment is, somewhat arbitrarily, fixed at $I^{\max} = 20$ (m^2). The value of the stock externality, corrected for the number of grazing animals, is finally given as $w = 20$ (EUR/animal).

5.2 Optimal steady states

Table 3 demonstrates the steady-state results in the private optimisation problem where the number of animals and capital are found as the solution to equations (10′) and (11′), and harvest and investment from equations (12) and (13), respectively. The results with the baseline parameter values are shown in the first column, while the next column indicates the effects of a 50 per cent increase in the meat price, to 360 (EUR/animal), while all the other parameters are kept at their baseline values. In the last column, the discount rate is increased by 50 per cent, to $\delta = 0.06$ while all the other parameters are kept at their baseline values. The steady state is interior all the time as the depreciation is below the investment constraint. In the baseline calculation, the optimal animal stock is 120 (animals), the capital stock becomes 200 (m^2) while the yearly profit is about EUR 9,800. The yearly investment reads $I^* = \gamma K^* = 8 < I^{\max}$ (m^2). The change in the discount rate has a modest impact on the optimal steady-state animal stock level while the effect on the

Table 3. Steady-state private optimal solution

		Baseline	p up 50%	δ up 50%
Animal stock (# of animals)	X^*	120	192	112
Capital (m^2)	K^*	200	322	168
Harvest (# of animals)	H^*	84	134	78
Investment (m^2)	I^*	8	13	7
Capacity utilisation (# of animals/m^2)	X^*/K^*	0.60	0.60	0.67
Yearly profit (EUR)	π^*	9,824	24,241	9,571

Table 4. Steady state social optimal solution and policy instruments

		Social planner, $w = 20$	$s^X = 20$	$s^p = 30.3$	$s^c = 21$
Animal stock (# of animals)	X^{**}	138	138	138	122[a]
Capital (m^2)	K^{**}	231	231	231	231
Harvest (# of animals)	H^{**}	97	97	97	85[a]
Investment (m^2)	I^{**}	9	9	9	9
Capacity utilisation (# of animals/m^2)	X^{**}/K^{**}	0.60	0.60	0.60	0.55[a]
Yearly profit (EUR)	π	–	12,689	12,856	10,139
Yearly subsidy (EUR)		–	2,756	2,922	194

[a]Not socially optimal.

capital stock is somewhat more substantial (see also the comparative static results Table 1). The yearly profit is only modestly affected. The 50 per cent slaughter price change, on the other hand, strongly affects the profit which is more than doubled compared with the baseline case, both because of the direct effect of increased per animal income on profit, but also because the farmer optimally adjusts the production to take further advantage of the price increase. The direct effect is most important quantitatively, as can be calculated from Table 3. Note also that capacity utilisation is unaffected when the price shifts up. The reason is that the investment schedule (11′) is linear and not affected by the slaughter price. A change in the discount rate, on the other hand, shifts this schedule as well as schedule (10′) and hence the capacity utilisation is changed.

Table 4 demonstrates the steady-state social planner solution and the effects of policy instruments. The first column indicates the planner solution for the given value of the stocking externality, $w = 20$ (EUR/animal). The animal stock increases about 15 per cent compared with the private solution, $X^{**} = 138$, and accordingly also the capital requirement in the same amount as the capacity utilisation stays unchanged. The effects of the animal stock subsidy $s^X = w$ are indicated in column 2 and demonstrate that the animal target X^{**}, but also the capital target K^{**} and therefore also H^{**} and I^{**}, are

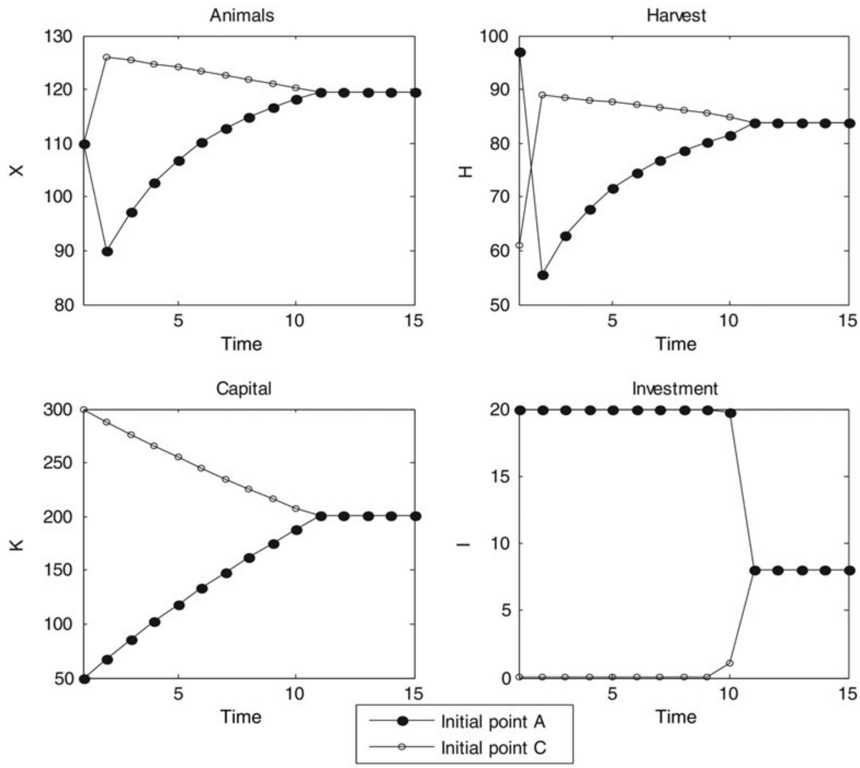

Fig. 2. Optimal approach paths baseline parameter values, private optimisation problem. Initial situation A ($K_0 = 50$, $X_0 = 110$) and C ($K_0 = 300$, $X_0 = 110$).

reached with this policy instrument alone. The yearly subsidy with this policy instrument amounts to EUR 2,756 and the steady-state farm profit becomes $\pi =$ EUR 12,689. In column 3, we find the effects of the meat price subsidy $s^p = w/(r - \delta)$ and where both targets X^{**} and K^{**} are reached with somewhat higher costs. Therefore, the farm profit now becomes higher than in the previous case and reads EUR 12,856. Finally, column 4 demonstrates the working of the capital subsidy instrument s^c scaled such that the social optimal capital stock is achieved, but accordingly not the optimal number of animals. The yearly subsidy becomes substantially lower than in the previous cases and accordingly also the steady-state farm profit becomes substantially lower.

5.3 Dynamics

The dynamics of the private optimisation problem are demonstrated in Figures 2 and 3 where the panels to the left are for the state variables while the panels to the right depict the corresponding harvest and investment paths. Four different initial situations are considered that correspond roughly to the initial states

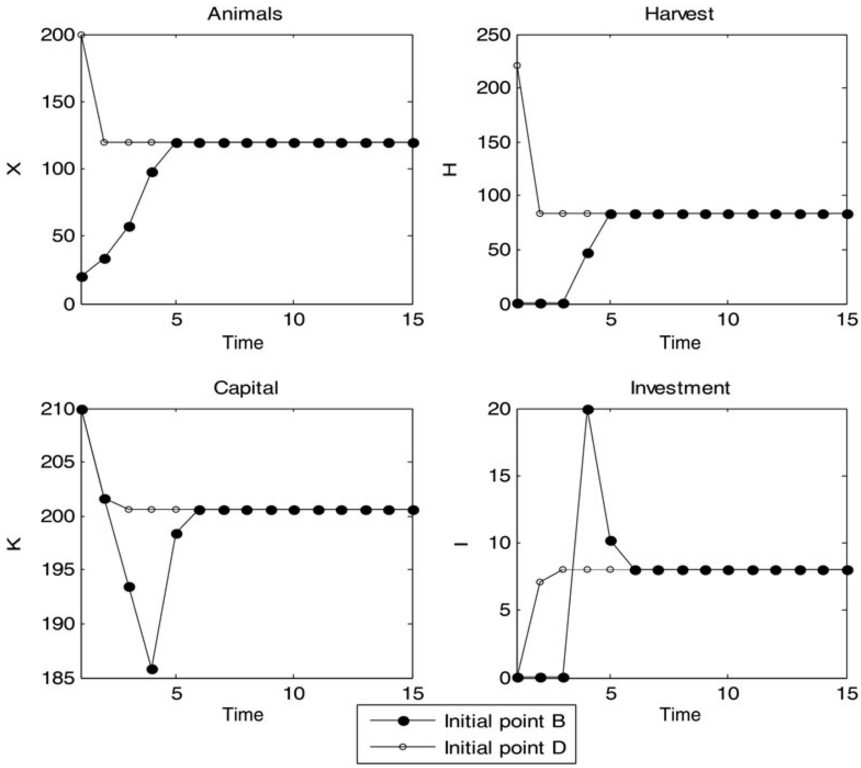

Fig. 3. Optimal approach paths baseline parameter values, private optimisation problem. Initial situation B ($K_0 = 210, X_0 = 20$) and D ($K_0 = 210, X_0 = 200$).

depicted in Figure A1 in the Appendix, such that (K_0, X_0) assumes the values (50, 110) at point A, (210, 20) at point B, (300, 110) at point C and (210, 200) at point D. The significance of these different initial situations is discussed in the Appendix.

In Figure 2, the approach paths from the initial points A and C are depicted. In both these situations the animal stock size is 110 (animals), which is close to the steady-state optimum (Table 3). The initial capital stock is either far below (A), or well above (C) its steady-state level. This figure illustrates the possibilities for the animal stock to over- or undershoot the steady state. If $K_0 = 50$, an immediate harvesting down of the animal stock is followed by a combination of maximum investment and singular harvest until the equilibrium is reached after about 12 years. Since the investment constraint is binding along the approach path, but not in the steady state, the different trajectories correspond to the ones depicted in Figure A1b. From $K_0 = 300$, both controls are set to zero and the animal stock grows above its steady-state level before the singular harvest schedule is followed, with zero investment.

Figure 3 demonstrates the optimal paths in the private optimisation problem when the initial states are fixed at points B and D. With $K_0 = 210$ in both cases, the initial capital stock is hence close to the optimal steady state (again, see Table 3), while the animal stock level is either far below (initial situation B with $X_0 = 20$), or far above (D with $X_0 = 200$), the optimal steady state. In either case, the steady state is reached faster than in the previous cases demonstrated in Figure 2, as both growth and decline in the animal stock is taking place faster than that of the capital stock. Initial point B entails zero harvest and investment, followed by one period of maximum investment (Region II in Figure A1b, see the Appendix) before the equilibrium is encountered. From initial situation D, impulse harvest and depreciation of the capital stock leads to the equilibrium after just two time periods. From point B, the capital stock undershoots the steady state (but may never overshoot, as discussed in Section 3.3 and the Appendix).

6. Concluding remarks

In this paper we have, from a theoretical point of view, analysed the dynamic optimisation problem of a profit maximising farmer who possesses both animals and man-made capital. The model builds on existing studies from the fisheries literature, but the important difference is that while capital is related to harvesting effort in the fisheries, capital contributes to production capacity to keep the animal stock during the winter in our farm model. The linearity of the model allows an intuitive graphical description that is rare in multi-dimensional optimisation problems. Steady states and optimal approach paths have been characterised analytically, and demonstrated by a numerical example related to Nordic sheep farming. Both the private and the social planner solution have been considered and where the amenity value of the cultural landscape (‘multifunctional agricultural production’) is included in the social planner problem. While exemplified by the Nordic sheep farming system, our model and reasoning has also direct relevance to sheep farming other places with a crucial distinction between the outdoor grazing season and the indoor winter season (e.g. mountain areas in France and Spain).

The steady state in the private solution is shown to be either an interior optimum with interior controls, or a constrained optimum with investment set to its maximum value (Result 1). The effects of parameter changes were studied analytically. We found that with a higher meat price the farmer will find it beneficial to increase the stock of animals as well as the amount of capital in the interior steady state (Result 2), while an increase in the discount rate yields opposite effects (Result 3).

As the objective function is linear in both control variables, the approach path is a combination of bang–bang and singular controls, and along the approach path at most one of the controls is singular. The dynamics are different from what is found in the typical fishery models, as in particular there will be a gradual building up of capital, not a one-time impulse investment where the capital stock overshoots the steady state. With capital, only undershooting is possible (Result 4). The animal stock may, on the other hand, both over- and under-shoot the optimal steady state (Result 5). In general, one of the controls will be singular along the approach path while the other is set to zero, if the upper investment constraint does not bind (Result 6).

In the social planner solution, the animal stock of the individual farmer represents a public good value because of landscape preservation. The steady state is characterised when internalising this stock externality, and we find that this solution demands more animals, in addition to more capital, compared with the private solution. Possible policy instruments are studied to achieve social optimality, and we find that both a meat price subsidy and an animal stock subsidy can reach the optimal number of animals as well as the optimal capital requirement. However, we find that the stock subsidy is more efficient than the meat price subsidy as an identical steady-state allocation can be reached with lower costs for the regulator (Result 7). An alternative policy instrument not considered here could have been per hectare grazing payment scheme. However, as land is not included in our model an assessment of this type of policy instrument is outside the scope of our analysis. Numerical examples illustrate the various results.

In both the private and social solution, we have focused on situations with a unique interior equilibrium. However, with different specifications of the cost function there may be several equlibria. With a positive discount rate, the choice of steady state will then in general depend on the initial situation, so that the system is history dependent. The dynamics of such a system will be a rather straightforward generalisation of the system analysed here however, once the optimal steady state is identified. Another possible extension is to include an absolute limit on the number of animals per square metre of housing, typically set by authorities to secure animal health. If this constraint binds along the approach path it will imply maximum investment together with positive harvest of animals. If the capacity utilisation constraint is binding in the steady state, it will imply singular steady-state harvest along with maximum investment.

The main contribution of this paper is related to the role of capital which is used here for maintaining the animals and hence plays no role in the harvesting process. In addition, we assume a domestic animal stock where the unit harvest cost is stock independent, and natural growth is density independent and hence also unaffected by stock size. Given that these assumptions also are valid in other types of production involving domestic renewable resources, the model here may have wider applications. Possible examples include other forms of livestock management and other areas of modern agricultural production, as well as aquaculture. The model may also possibly be extended into management of other types of natural resources, like terrestrial wildlife.

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Appendix

Optimal approach paths

As indicated in Section 3.3, the optimal trajectories result from a combination of extreme and singular controls. We know that both controls can be singular simultaneously only at an interior steady state, so that one of the control constraints must always bind outside an equilibrium. Whenever the animal stock is above the H^S-schedule, it will be harvested down instantaneously (or more precisely, within one-time period), until the H^S-schedule is reached since H_t is unconstrained from above. Then either (i) the system will follow the H^S-schedule, with singular harvest for a period of time, or (ii) harvest is set to zero, in which case the H^S-schedule acts as a switch between extreme controls. Ignoring the case where $H_t > H^S$, which is impossible for any more than one-time period, we now consider the various alternative control regimes. The different cases can be best understood with reference to Figure A1 where a situation with a unique interior equilibrium is depicted. The singular control schedules are in accordance with the specific functional forms used in Section 5 [equations (10′) and (11′)]. Four different initial states, labelled A, B, C and D, are shown along with the optimal approach paths originating from them. Note that initial states A and C have the same value for the animal stock, whereas initial states B and D represent the same capital stock value. These properties are further exploited in the Section. Figure A1a demonstrates the first three cases, where the upper investment constraint does not bind along the approach path.

Case 1: $H_t = 0$, $I = I^S$. The only possibility when investment is singular outside of the steady state is that harvest is zero. This happens when, as from an initial situation such as A or B, the initial capital stock is below the steady-state level and the system has been controlled to reach the singular investment schedule. The system will then follow the I^S-schedule (11) towards the steady state, when the investment constraint does not bind.

Case 2: $H = H^S$, $I_t = 0$. From an initial situation such as point D where both stocks are above their interior steady-state levels, the animal stock is harvested down until the H^S-schedule (10) is reached, and the system moves leftwards along the H^S-schedule towards the equilibrium. Note that the I^S-schedule plays no role here, and is therefore represented by a dashed line in the figure.

Case 3: $H_t = 0$, $I_t = 0$. Here both controls are set to zero, which happens when the state of the system is below both singular control schedules, as at point C. This control regime continues until one of the two singular control schedules is reached, and one of the two above alternatives takes over.

The next two 'intermediate' cases, where the upper investment constraint prevents the system from following the I^S-schedule, are shown in Figure A1b. Both these cases depend on the system having reached either the H^S-schedule or the I^S-schedule below and to the left of the equilibrium. This may happen if the initial states are given by points A or B.

Case 4: $H_t = 0$, $I_t = I^{\max}$. When $H^S < 0$, meaning that following the H^S-schedule would require restocking of animals, which is omitted in our

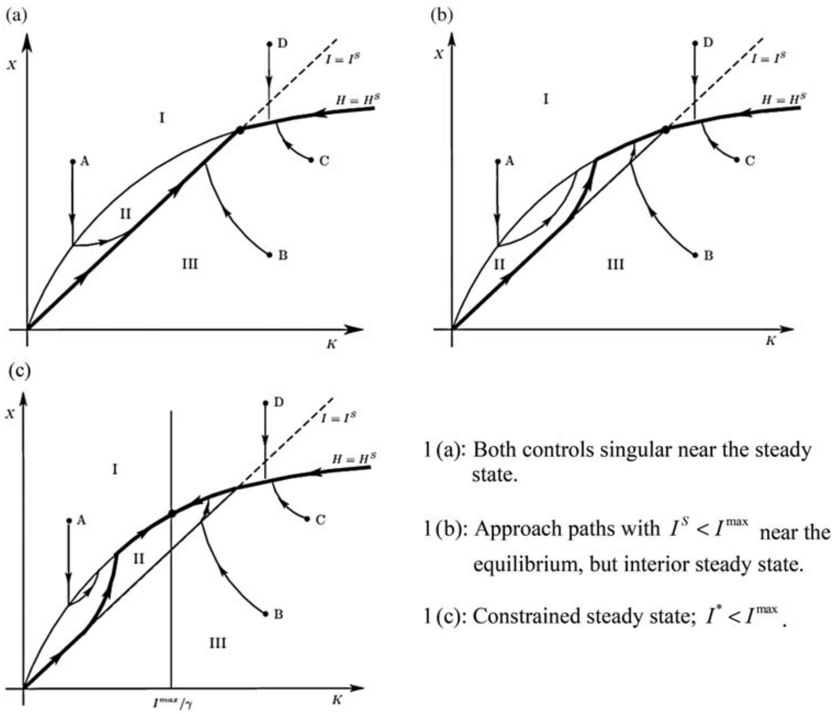

Fig. A1. Optimal approach paths, unique steady state. Private optimisation problem.

model, and $I^S > I^{\max}$, so that the maximum investment constraint does not allow the system to follow the I^S-schedule either, the state of the system will be somewhere below the H^S-schedule and above the I^S- schedule.

Case 5: $H = H^S, I = I^{\max}$. This situation arises when the system has reached the H^S-schedule (9), either after an initial impulse harvest, or from a situation such as in case 4, but the maximum per period investment is not sufficiently large to detract the system from the H^S-schedule. The system will then follow the H^S-schedule to the steady state.

The last situation to consider is the alternative steady state where the upper investment constraint is binding. Figure A1c demonstrates. The equilibrium can now be found as a point on the H^S- schedule, below and to the left of the intersection point, with investment set to its maximum level at every point in time, $I^* = I^{\max} < I^S$. As shown in the figure, the approach path is along the H^S-schedule from both directions in this case. However, for sufficiently low stock values it is still possible to follow the I^S-schedule, as the animal stock growth is a constant share of the animal stock size, whereas the maximum investment is assumed to be independent of the size of the existing capital stock.

The different control scenarios can be further characterised by dividing the state-state space into three regions, with three different transitional control regimes. Region I: above the H^S-schedule; impulse harvest, Region II: below the H^S-schedule, above the I^S-schedule; $H_t = 0, I_t = I^{\max}$ and Region III: below both schedules; $H_t = I_t = 0$.

As is evident from Figure A1, the singular control schedules act either as switch lines or approach paths, depending on the control constraints. The approach path is identified as a bold line, which in Figure A1a consists of the part of the I^S-schedule (11), that is, to the left of the optimal steady state, and the part of the H^S-schedule (10) that is to the right of the equilibrium. The upper constraint on investment may also entail that the H^S-schedule must be followed even from the left, at least when the equilibrium is sufficiently close. This situation is depicted in Figure A1b,c.

Whenever the initial point is above the singular harvest schedule, a situation exemplified by points A and D in Region I, the stock will be slaughtered down immediately until the H^S-schedule is reached. If the state of the system is now above the singular investment schedule, the H^S-schedule acts as a switch and harvest is set to zero, as is the case when starting from point A. If not, the rest of the approach path is along the H^S-schedule, as with the trajectory from point D. When starting from below both schedules, as from points B and C, the singular approach path is the one of the control schedules that is encountered first, after a period with zero harvest and investment. As indicated in Figure 1a, the approach path is thus the lower one of the two singular control schedules, when feasible. As seen on Figure 1b and c, however, a part of this control path may not be feasible if the upper investment constraint binds. In this case only the leftmost part of the I^S-schedule can be followed, while the equilibrium is encountered along the H^S-schedule from both directions. The I^S-schedule may then act partly as a switch between zero and maximum investment, which the trajectory from point B indicates.

As indicated in the main text (Section 3.3), the two first dynamic results regard the monotonicity of the approach paths, and are related to results from the fisheries literature. First observe that the approach path is monotonic with respect to both X and K along the singular control schedules and in Region II (see the discussion above). Also note that (i) in Region I there is only impulse harvest and no investment and (ii) in Region III the monotonic part of approach path may be encountered on either side of the equilibrium if $K_0 > K^*$ and $X_0 < X^*$. The first of these results is stated as Result 4 in the main text.

Proof of Result 4. Positive investment cannot occur in any of the two regions outside the monotonic part of the approach path. Hence, overshooting is impossible. Undershooting happens if, from Region III with $K_0 > K^*$ and $X_0 < X^*$, the monotonic part of the approach path is reached where $K_t < K^*$.

Proof of Result 5. When $X_0 > X^*$ and $K_0 < K^*$ in Region I, the animal stock will be reduced immediately until the monotonic part of the approach path is reached where$X_t < X^*$, implying undershooting. From Region III with $K_0 > K^*$ and $X_0 < X^*$, overshooting occurs if the monotonic part of the approach path is reached where $X_t > X^*$.