The dynamics of an animal - vegetation system:
Sheep farming

by

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Abstract
The paper studies the economy and ecology of sheep farming at the farm level in a Nordic context, with a crucial distinction between the outdoors grazing season and the winter indoors feeding season, and where climate conditions fix the length of the grazing season. Two different categories of animals, ewes (adult females) and lambs, and one plant species are included in our ecological model. The farmer is assumed to maximize present-value profit where the revenue is made up income from meat and wool production. We find that livestock cycles may represent an optimal management policy. We also show that in a possible steady state with a constant number of animals and constant vegetation quantity, slaughtering either only lambs or only ewes is optimal.

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1. Introduction

In this paper, we show that livestock cycles may represent an optimal policy at the farm level in an economic sheep-vegetation trade-off model. The main content of this trade-off is that high sheep densities yield high farm output in number of animals slaughtered. On the other hand, high sheep densities relative to pasture productivity cause a reduction in meat production per animal and thus a decrease in income per animal. We also show that, in a possible steady state with a constant number of animals and constant vegetation quantity, the harvesting decision is shaped by economic factors alone.

We consider a Nordic sheep farming system. Within this farming system, the individual farmer faces several decisions. The problem analyzed here is that of utilizing a given farm capacity (i.e., farm size) to provide the optimal number of animals to be fed and kept indoors during the winter season. A corollary of this problem is to assess the effect that summer grazing density has on vegetation productivity and on per-animal meat production. While we show that livestock cycles may represent an optimal solution to this problem, we also find that the economic benefit of an optimal cycle policy is typically small compared keeping a fixed number of animals. Our stocking problem has similarities with the standard predator-prey renewable natural resource problem (see, e.g., Clark 1990) where the sheep is the predator while the vegetation is the prey. However, our animal – vegetation interaction is unidirectional as the vegetation quantity has no direct influence on the animal growth number. The link goes indirectly through the weight gain and hence, the value per animal.

Sheep is the main livestock in animal husbandry in the Nordic countries Norway and Iceland, and most of the cultivated land is used for winter fodder production (58% and 95% in Norway and Iceland, respectively; see e.g., Austrheim et al. 2008) In these countries, there is a crucial distinction between the outdoors grazing season (spring, summer and fall) and the indoors winter feeding period, which includes when the lambs are born in late winter to early spring, just before the grazing season starts. This distinction over the year cycle is also crucial in our analysis, and where the animals are structured in two categories, lambs and adult females (ewes). This age structured model is extended to take into account that the outdoors grazing conditions generally represent

a constraint on the animal weight growth. Climate conditions fix the length of the grazing season in our Nordic farming context.

Although lags have been used to study the adjustment of the breeding inventory of sheep (see, e.g. Deese 2003), sheep technology exhibits very low delays. The gestation birth delay is between 145 and 153 days, and maturation for slaughter takes does not normally exceed 130 days (Austrheim et al. 2008). Therefore, we assume that gestation, birth and maturation takes place within the same period of one year. Because sheep farming is a managed system, animal growth is assumed density independent and hence linear. On the other hand, vegetation growth, as well as lamb animal weight growth, is nonlinear. Although the number of animal growth is linear without lags, the interaction between animals and vegetation introduces a second-order lag in our problem. This special feature suggests that stationary solutions might be cyclical. That is to say, sustainable management solutions can either be steady states where vegetation and animals remain constant through time or exhibit periodical cycles where vegetation and the number of animals fluctuate around constant values every second year. Therefore, our animal vegetation system induces dynamics that can be similar to the *cattle cycles* induced by maturity lags (see, e.g., Rosen 1987 and Rosen et al. 1994). There are numerous papers, mostly in the ecology literature, some also in the range management literature, that examine sheep grazing. The present paper builds on Skonhoft et al. (2010) who analyzed the sheep – vegetation system in ecological equilibrium. In contrast to this, the dynamics is at the focus here. The present paper has some similarities with Torell et al. (1991), Huffaker and Cooper (1995) and Finnoff et al. (2008) which all highlight the importance of the tradeoff between numbers of grazers and the weight gain of the animals.

The paper is organized as follows. In the next section two, we first present and discuss briefly the Nordic sheep farming system and describe the ecological model, the cost, and the revenue functions. In section three, the optimal program is formulated while the optimal slaughtering policy is described in section four. In section five, we proceed to show under what condition cycles are optimal. Section six provides a numerical illustration, while section seven summarizes and concludes our findings.

2. The model

The following analysis is related to economic and ecological conditions found in Norway, but these also exist in Iceland. There are approximately 16,000 sheep farms in Norway, all family farms, with around 2.1 million grazing animals during the outdoors grazing season. Norwegian farms are located either close to mountain areas and other sparsely populated areas or along the coast. The main product is meat, which accounts for about 80% of the average farmer's income. The remainder comes from wool, because sheep milk production is virtually nonexistent today (Nersten et al. 2003). Housing and indoor feeding is required throughout winter because of snow and harsh weather conditions. Lambs are born from late winter to early spring. During late spring and early summer, the animals usually graze on fenced land close to the farm. When weather conditions permit sheep are released into rough grazing areas in the valleys and mountains. The outdoors grazing season typically ends middle of September. The animals are then mustered and the wool is shorn. Slaughtering takes place immediately or after a short period of grazing on the farmland. See Figure 1.

Figure 1 about here

The ecological model is just as in Skonhoft et al. (2010) and comprises an animal growth equation, a vegetation growth relationship, and an equation defining how the food intake during the outdoors grazing season influences the animal weight gain and slaughter weight. The number of animals influence the vegetation growth during the grazing season, but not the *vice versa* as mortality and fertility rates are assumed to be fixed. The population growth is thus linear; that is, more competition among the individuals for the available resources when the population increase is assumed not to cause increasing mortality rates and decreasing fecundity rate. Linear population growth is typical in a managed animal system (see, e.g., the cattle model of Rosen et al. 1994). On the other hand, more animals and higher animal density are assumed to slow down the individual growth rates represented by the weight of the animals. Just as in Huffaker and Wilen (1991) and others this mechanism works through the vegetation growth equation. The ecological model is formulated at a discrete time with a time resolution of one year, and with a seasonal subdivision between the outdoors grazing period (spring, summer and fall) and indoors winter feeding period (Figure 1). The sheep population is structured as adult females (ewes) and lambs because all male lambs are slaughtered since very few (or none when artificial insemination is practiced) are kept for breeding.

Therefore, only female adults are considered. The population is measured in fall after slaughtering. In the single period of one year, three events occur in the following order: First, recruitment and lambing, then natural mortality, and finally slaughtering. Therefore, when neglecting the small natural mortality during the indoors season, all mortality is assumed to take place during the outdoors grazing season related to various diseases and accidents, and in some regions also to large predators (see, e.g., Ekspertutvalget 2011). Natural mortality is larger for lambs than for adults.

We start by formulating the animal growth equation where the number of adult females in year $(t+1)$, X_{t+1}, after the slaughter consists of the previous year's adults and female lambs that have survived natural mortality during the outdoors grazing seasons and have not been slaughtered. This is written as $X_{t+1} = Y_t s^y (1-h_t^y) + X_t s^x (1-h_t^x)$, where Y_t is the number of female lambs in year t, s^x and s^y are the constant annual natural survival rates of adult females and lambs, respectively, and $0 \le h_t^x \le 1$ and $0 \le h_t^y \le 1$ are the fractions slaughtered. With the constant annual fecundity rate b (lambs per adult female) and ψ as the fraction of female lambs recruited and no natural mortality during the winter, $Y_t = \psi b X_t$ yields the number of female lambs. Therefore, the ewe population growth reads:

(1) $\qquad X_{t+1} = \psi b X_t s^y (1-h_t^y) + X_t s^x (1-h_t^x)$.

Because the population growth equation (1) is linear for number of animals, there are infinite combinations of harvesting fractions that may sustain a stable (or steady state) population. Therefore, for a constant number of animals $X_{t+1} = X_t = X$, we have:

(1') $\qquad X = \psi b X s^y (1-h^y) + X s^x (1-h^x)$,

or simply $1 = \psi b s^y (1-h^y) + s^x (1-h^x)$ when $X > 0$. This equation also defines the highest possible equilibrium harvesting rates with $h^y = [1-(1-s^x)/\psi b s^y] < 1$ for the lambs when $h^x = 0$, and $h^y = \min\{1,[(1-(1-\psi b s^y)/s^x]\}$ for the ewes when $h^y = 0$.

We assume composite homogeneous vegetation, measured in tons of vegetation biomass, and given as V_t in year t. The vegetation growth function follows the Noy-Meir (1975) model (but Noy-Meir uses continuous time) and consists of natural growth

minus animal consumption in which *per-animal* vegetation consumption increases with vegetation availability, but at a decreasing degree:

(2) $\quad V_{t+1} - V_t = f(V_t) - g(V_t)(1+b)X_t$.

It is assumed that the vegetation consumption, and hence the vegetation pressure, is governed by the number of animals in the beginning of the grazing season, $(1+b)X_t$ and with similar food intake of ewes and lambs. $g(V_t)$ is thus the per-capita consumption function, assumed to be increasing and strictly concave, while $f(V_t)$ yields the natural growth function, assumed to be one-peaked and strictly concave. These functions are scaled such that the consumption curve intersects with the natural vegetation growth curve from below; that is, $f'(V) < g'(V)(1+b)X$ holds at the unique (interior) equilibrium. In the numerical illustrations, we use the standard logistic natural growth function indicating that the vegetation quantity in principle may approach zero. However, overgrazing is not a serious problem, and studies of productive and species-rich alpine environments typically show small and moderate effects of grazing on plant community patterns (Austrheim et al. 2008). For this reason, the outdoors grazing conditions are assumed not to influence sheep fertility and mortality.

While the vegetation quantity has no influence on the number of animal growth, it feeds back on the animals by influencing the weight of the animals. The weight gain of the lambs during the grazing season coincides with the weight at the end of the season; that is, the slaughter weight (kg per animal). It is assumed proportional to per-animal vegetation consumption:

(3) $\quad w_t^y = qg(V_t)$,

where the parameter $0 < q < 1$ translates grazing biomass into meat biomass. For the adults, there is generally no weight change during the grazing season (Austrheim et al. 2008), and the adult slaughter weight is hence fixed and determined outside the model:

(4) $\quad w_t^x = w^x$.

In the above described predator – prey system, there is only harvest of the predator, the animals. In addition, because of the linear animal growth relationship, we find that the amount of vegetation has no direct influence on the number for animal growth. However, there is an ecological and economic link through the weight gain of the

animals, such that better grazing conditions, *ceteris paribus*, increase the per lamb economic value. Slaughtering has no direct effect on the vegetation. However, slaughtering year $(t-1)$ determines the number of animals the next year t and thus indirectly the amount of vegetation in year $(t+1)$. This special feature has an important implication for the dynamic behavior of our system. Therefore, although the number of animal growth is linear without lags, the interaction between animals and vegetation introduces a second-order lag in the problem and suggests that stationary solutions of our model might be cyclical. That is to say, sustainable management solutions can either be steady states, where vegetation and animals remain constant through time or exhibit periodical cycles, where vegetation and the number of animals fluctuate around constant values every second year. Therefore, our animal vegetation system can cause dynamics similar to the *cattle cycles* induced by maturity lags (see also introductory section).

The revenue of the farmer is made up of income from meat and wool production. Because slaughtering takes place after natural mortality, the number of ewes and female lambs removed in year t are $X_t s^x h_t^x$ and $\psi b X_t s^y h_t^y$, respectively. As mentioned above, the entire male lamb subpopulation $(1-\psi) b X_t s^y$ is slaughtered. With p^x as the ewes slaughtering price (NOK per kg) and p^y as the lamb slaughtering price, both assumed to be constant over time and independent of the number of animals supplied at the farm level and net of slaughtering costs, the current meat income of the farmer reads

$$M_t = [p^y w_t^y b X_t s^y (\psi h_t^y + 1 - \psi) + p^x w_t^x X_t s^x h_t^x].$$

The other income component is wool sale where it is assumed that the adult animals are shorn two times a year, in the spring and in the autumn just before slaughtering, while the lambs are shorn just before slaughtering. The yearly wool income may then written as $W_t = z[\sigma_s X_t + \sigma_a s^x X_t + \tau b s^y X_t]$, where z is the net (net of shearing costs) wool price (NOK per kg wool), σ_s and σ_a are the (average) per unit adult spring and autumn outputs (kg per animal), respectively, and τ is the per lamb output. This expression may be simplified to $W_t = z\theta X_t$, such that $\theta = \sigma_s + \sigma_a s^x + \tau b s^y$ is the demographic and seasonally adjusted per unit wool output coefficient.

As already indicated, we assume a given farm capacity, and include only the indoors season costs as the outdoors costs are rather small. These indoors costs, including fodder, electricity and veterinary costs, are assumed to depend on the size of the winter population $C_t = C(X_t)$, with $C(0) = 0$, $C' > 0$ and $C'' \geq 0$. Therefore, when ignoring discounting within the year and inserting equations (3) and (4), the yearly profit of the farmer is described by $\Pi(X_t, V_t, h_t^y, h_t^x) = M_t + W_t - C_t$

$$= p^y q g(V_t) b X_t s^y (\psi h_t^y + 1 - \psi) + p^x w^x X_t s^x h_t^x + z\theta X_t - C(X_t).$$

3. The optimal program

The farmer is assumed always to be 'rational' and well informed with the goal of maximizing present-value profit $\sum_{t=0}^{\infty} \beta^t \Pi(X_t, V_t, h_t^y, h_t^x)$, subject to the animal growth condition (1) and the vegetation growth condition (2), where $\beta = 1/(1+\delta)$ is the discount factor and $\delta \geq 0$ is the (yearly) fixed discount rent. In addition, the initial number of animals and the initial condition of the vegetation quantity are given; that is, X_0 and V_0 are known. The Lagrangian of this problem may be written as

$$L = \sum_{t=0}^{\infty} \beta^t \{ [p^y q g(V_t) b X_t s^y (\psi h_t^y + 1 - \psi) + p^x w^x X_t s^x h_t^x + z\theta X_t - C(X_t)]$$

$$- \lambda_t [X_{t+1} - \psi b X_t s^y (1 - h_t^y) - X_t s^x (1 - h_t^x)] - \mu_t [V_{t+1} - V_t - f(V_t) + g(V_t)(1+b)X_t] \},$$

where $\lambda_t > 0$ is the animal resource shadow price and $\mu_t > 0$ is the vegetation resource shadow price. Following the Kuhn-Tucker theorem, the first order necessary conditions (assuming $X_t > 0$ and $V_t > 0$) are:

(5) $\quad \beta^{-t} \partial L / \partial h_t^y = \partial \Pi / \partial h_t^y - \lambda_t \psi b s^y X_t \overset{\leq}{\underset{>}{=}} 0 \; ; \; 0 \leq h_t^y \leq 1$,

(6) $\quad \beta^{-t} \partial L / \partial h_t^x = \partial \Pi / \partial h_t^x - \lambda_t s^x X_t \overset{\leq}{\underset{>}{=}} 0 \; ; \; 0 \leq h_t^x \leq 1$,

(7)

$$\beta^{-t} \partial L / \partial X_{t+1} = \beta \partial \Pi / \partial X_{t+1} + \beta \lambda_{t+1} [\psi b s^y (1 - h_{t+1}^y) + s^x (1 - h_{t+1}^x)] - \beta \mu_{t+1} g(V_{t+1})(1+b) - \lambda_t = 0$$

and

(8) $\beta^{-t} \partial L / \partial V_{t+1} = \beta \partial \Pi / \partial V_{t+1} + \beta \mu_{t+1} [1 + f'(V_{t+1}) - g'(V_{t+1})(1+b)X_{t+1}] - \mu_t = 0$,

and where $\partial\Pi/\partial h_t^y = p^y qg(V_t)bs^y\psi X_t$, $\partial\Pi/\partial h_t^x = p^x w^x s^x X_t$,

$$\partial\Pi/\partial X_{t+1} = [p^y qg(V_{t+1})bs^y(\psi h_{t+1}^y + 1 - \psi) + p^x w^x s^x h_{t+1}^x + z\theta - C'(X_{t+1})] \text{ and}$$

$$\partial\Pi/\partial V_{t+1} = p^y qg'(V_{t+1})bX_{t+1}s^y(\psi h_{t+1}^y + 1 - \psi).$$

Control condition (5), which reads $(p^y qg(V_t) - \lambda_t)X_t \overset{\leq}{\underset{>}{=}} 0$ when inserted for $\partial\Pi/\partial h_t^y$,

states that the slaughtering of the lambs should occur at the point where the per animal slaughter value is below, equal to or above the cost of reduced growth in stock numbers, evaluated at the animal shadow price. Control condition (6) for the adults reads

$(p^y w^x - \lambda_t)X_t \overset{\leq}{\underset{>}{=}} 0$ when inserting for $\partial\Pi/\partial h_t^x$, and is analogous. The animal stock

(portfolio) condition (7) indicates that the number of animals should be maintained so that its shadow value equalizes the marginal meat value plus the contribution to further animal growth, evaluated at the animal shadow price and taking discounting into account, minus the marginal grazing cost, evaluated at the grazing shadow price, and taking discounting into account as well. In a similar manner, the vegetation stock (portfolio) condition (8) states that the vegetation shadow price should equalize the marginal profit gain through a higher lamb weight plus its indirect effect through the vegetation quantity, evaluated at the vegetation shadow price, and taking into account discounting.

From the control conditions (5) and (6), it is evident that the per animal slaughter value steers the slaughtering composition. Assume first that the vegetation quantity is 'high' so that the per animal slaughter value of the lambs exceeds that of the ewes, $p^y qg(V_t) > p^x w^x$. This is the typical situation (see below) and because of the Kuhn-Tucker conditions it indicates a higher harvesting rate of the lambs than the ewes, which can be satisfied in three ways: i) $h_t^y = 1$ and $0 < h_t^x < 1$, ii) $h_t^y = 1$ and $h_t^x = 0$ and iii) $0 < h_t^y < 1$ and $h_t^x = 0$. On the contrary, with heavy grazing pressure and 'low' vegetation quantity, such that $p^x w^x > p^y qg(V_t)$, and hence more aggressive harvesting of the adults, we find that the control conditions can be satisfied either as iv) $h_t^x = 1$ and $0 < h_t^y < 1$, v) $h_t^x = 1$ and $h_t^y = 0$, or as vi) $0 < h_t^x < 1$ and $h_t^y = 0$. Therefore, the lamb slaughtering mortality will always exceed the ewes slaughtering mortality when the

meat value (NOK per animal) is highest for the lambs. However,

when $p^x w^x > p^y qg(V_t)$, the ewes slaughtering mortality will be highest.

4. Optimal slaughtering

In a possible (interior) steady state where all variables are constant over time with a 'high' vegetation quantity and $p^y qg(V) > p^x w^x$ (the time subscript is dropped when considering steady state) , we find that the above control conditions can be satisfied only as possibility iii), $0 < h^y < 1$ and $h^x = 0$, because slaughtering all the lambs as already indicated is not an option in a possible steady state. A corollary of $h^x = 0$ is that lamb slaughtering should take place at the highest level compatible with the sheep population equilibrium; that is, $h^y = 1 - (1 - s^x) / \psi bs^y < 1$, cf. equation (1') and section two.

In the opposite case of a 'low' vegetation quantity and more valuable ewes than lambs, $p^x w^x > p^y qg(V)$, the control conditions in a possible steady state can generally be satisfied either as the above cases iv), v), or vi). However, while steady state slaughtering of all adults may be an option in principle, it will not happen because of the actual demographic parameter values (numerical section seven and Table A1). Therefore case vi) with $h^x = 1 - (1 - \psi bs^y) / s^x < 1$ and $h^y = 0$ (section two) will be the only steady state possibility when adults are more valuable than lambs. This is stated as:

Proposition 1. In a possible steady state where all variables are constant over time with a 'high' vegetation level, slaughtering only lambs is optimal, and this should take place at the highest level compatible with population equilibrium determined by sheep biological factors alone. In a possible steady with a 'low' vegetation level, slaughtering only adults is optimal and this should again take place at the highest level compatible with population equilibrium, also determined by biological factors alone.

The result of only one animal harvesting category being optimal has similarities with the well-known finding in Reed (1980) who studied the maximum sustainable yield problem of a fishery. See also the extension in Skonhoft et al. (2012). On the other

hand, the reason for slaughtering at the highest level compatible with ecological equilibrium, either of lambs or adults, follow from the assumption of constant natural mortality rates and constant fertility rate and hence lack of any density-dependent effects in the animal growth equation (1). At the same time, this means that the possible optimal steady state harvesting rate, in contrast to what is found in most (if not all) wild animal (and fish) bioeconomic models, depends only on animal biological parameters (fertility and mortality). The working of these factors are straightforward, as higher fertility as well as lower mortality yield higher harvesting rates in both of the above cases iii) and vi).

It is also possible to say something about the dynamics and transitional paths. Along the transition paths, we may state:

Proposition 2. Assume that $p^y qg(V_t) > p^x w^x$. If $X_{t+1}/X_t \geq s^x$, the optimal policy is given by $h_t^x = 0$ and $h_t^y = 1 - \left[\dfrac{(X_{t+1}/X_t) - s^x}{b\psi s^y} \right]$. If $X_{t+1}/X_t < s^x$, $h_t^y = 1$ and

$h_t^x = 1 - \dfrac{(X_{t+1}/X_t)}{s^x}$ *are optimal. In the opposite case of $p^x w^x > p^y qg(V_t)$, if*

$X_{t+1}/X_t > b\psi s^y$, *the optimal policy is given by $h_t^y = 0$ and $h_t^x = 1 - \dfrac{(X_{t+1}/X_t) - b\psi s^y}{s^x}$.*

Otherwise, if $X_{t+1}/X_t < b\psi s^y$, $h_t^x = 1$ and $h_t^y = 1 - \dfrac{(X_{t+1}/X_t)}{b\psi s^y}$ are optimal.

Proof: See Appendix 1.

Proposition 2 indicates that, along transitional paths, the optimal harvest policy no longer depends only on biological factors. The harvest policy is also contingent upon economic forces working through the number of animals. Furthermore, if the optimal long-term solution is not a steady state, we also find that the optimal harvest policy no longer depends only on biological factors.

5. Pulse slaughtering cyclical equilibrium
In the previous section, we studied transitional paths and some properties of a possible steady state where we found a stationary solution with a constant number of animals, vegetation level and slaughtering rates through time. We now ask if other stationary

solutions may be present. To answer this question, we need to check if there exist periodic solutions.

Periodic solutions generalize the concept of steady state. A periodic solution may arise when a steady state is unstable. The stability of our animal-vegetation system can be studied through the Jacobian matrix. The Jacobian matrix of the system (1) and (2) is given by

$$J = \begin{pmatrix} 1 + f'(V_t) - g'(V_t)(1+b)X_t & -g(V_t)(1+b) \\ 0 & \psi bs^y(1-h_t^y) + s^x(1-h_t^x) \end{pmatrix}.$$ In any steady state, we

have $1 = \psi bs^y(1-h_t^y) + s^x(1-h_t^x)$ and $f(V_t) - g(V_t)(1+b)X_t = 0$. Therefore, the eigenvalues of the animal-vegetation system at steady state must verify the equation

$$p(\lambda) = |J - \lambda I| = \begin{vmatrix} 1 + f'(V_t) - g'(V_t)\dfrac{f(V_t)}{g(V_t)} - \lambda & -g(V_t)(1+b) \\ 0 & 1 - \lambda \end{vmatrix} =$$

$(1 + f'(V_t) - g'(V_t)\dfrac{f(V_t)}{g(V_t)} - \lambda)(1 - \lambda) = 0$. We find that the Jacobian of the animal-

vegetation system has a single real eigenvalue in the unit circle with value +1. This demonstrates that any steady state of the animal vegetation system is not stable.

Another candidate for a stationary solution is a two-year periodic cycle. This type of solution is an orbit where $V_{t+1} = V_t + f(V_t) - g(V_t)(1+b)X_t$,

$V_t = V_{t+1} + f(V_{t+1}) - g(V_{t+1})(1+b)X_{t+1}$, $X_{t+1} = [\psi bs^y(1-h_t^y) + s^x(1-h_t^x)]X_t$ and

$X_t = [\psi bs^y(1-h_{t+1}^y) + s^x(1-h_{t+1}^x)]X_{t+1}$, with $V_{t+1} \neq V_t$ and $X_{t+1} \neq X_t$. In the fishing economics literature, with age structured models, this solution is known as pulse fishing, and it may become optimal because of imperfect fishing selectivity (see, e.g., Tahvonen 2009). We have perfect harvesting selectivity in our model, but the following proposition states that periodic cycles may be optimal.

Proposition 3. Assume that: i) there exists a steady state with $p^y qg(V) > p^x w^x$ and ii) the marginal maintenance cost is higher than the marginal wool income, $c \geq z\theta$, where

$C'(X) = c$. When $g'(V) > \beta\dfrac{f'(V)}{f(V)}g(V)$, a *pulse slaughtering cycle is then always more*

profitable than a steady state solution with constant number of animals and constant vegetation quantity.

Proof: See Appendix 1.

This proposition hinges on the assumption that the unit maintenance cost should be higher than the marginal wool income. This generally holds (see numerical section). In a periodic cyclical equilibrium, the vegetation natural growth function $f(V_t)$ does not necessary equal the sheep consumption $g(V_t)(1+b)X_t$. Therefore, by reducing the number of animals in the second period, the vegetation quantity can increase in the first period. That is, if we reduce the number of animals in period two, the profit in period one can increase because the weight per lamb, and hence the value per lamb, in period one increases. To increase one unit of vegetation in period one, it is necessary to reduce the number of animals with $f'(V)/f(V)$ units in period two. If the weight gain per lamb due to increased vegetation in period one, $g'(V)$, is greater than the reduction in the discounted reduced number of slaughtered animals in period two, $\beta \dfrac{f'(V)}{f(V)} g(V)$,

then pulse slaughtering is always better than a steady state solution with a fixed number of animals and constant vegetation quantity through time. Therefore, Proposition 3 demonstrates that, even when $p^y q g(V_t) > p^x w^x$, the steady state may not represent the optimal solution.

In what follows we will characterize the two-year periodic cycle equilibrium where $h_1^y = 1$, $0 < h_2^y < 1$ and $h_3^x = h_2^x = 0$. In a two-year stationary cycle, the conditions (5) - (8) take the form; $\partial \Pi / \partial h_1^y - \lambda_1 \partial n(h_1^y) / \partial h_1^y X_1 - \xi = 0$; $h_1^y = 1$,

$\partial \Pi / \partial h_2^y - \lambda_2 \partial n(h_2^y) / \partial h_2^y X_2 = 0$; $0 < h_2^y < 1$, $\partial \Pi / \partial X_1 + \lambda_1 n(h_1^y) - \mu_1 \partial V_2 / \partial X_1 - \beta^{-1} \lambda_2 = 0$,

$\partial \Pi / \partial X_2 + \lambda_2 n(h_2^y) - \mu_2 \partial V_1 / \partial X_2 - \beta^{-1} \lambda_1 = 0$, $\partial \Pi / \partial V_1 + \mu_1 \partial V_2 / \partial V_1 - \beta^{-1} \mu_2 = 0$,

$\partial \Pi / 2V_1 + \mu_2 \partial V_1 / \partial V_2 - \beta^{-1} \mu_1 = 0$, $X_1 = n(h_2^y) X_2$, $n(h_1^y) n(h_2^y) = 1$,

$V_1 = V_2 + f(V_2) - g(V_2)(1+b)X_2$ and $V_2 = V_1 + f(V_1) - g(V_1)(1+b)X_1$. The slack multiplier ξ is positive because the feasibility constraint $0 \leq h_t^y \leq 1$ is binding at the upper limit, and because of the two-year cycle; that is, we have $X_3 = X_1$, $V_3 = V_1$,

$X_4 = X_2$, $V_4 = V_2$, and so forth. Note that have simplified the notation by using the animal growth rates along the cycle, $n(h_1^y)$ and $n(h_2^y)$, and by defining

$$V_1(V_2, X_2) = V_2 + f(V_2) - g(V_2)(1+b)X_2 \text{ and } V_2(V_1, X_1) = V_1 + f(V_1) - g(V_1)(1+b)X_1.$$

For computing the pulse slaughtering cyclical equilibrium, we set $h_1^y = 1$, $n(h_1^y) = s^x$, $n(h_2^y) = 1/s^x$, $h_2^y = \left[\psi b s^y + s^y - (1/s^x) \right] / \psi b s^y$ and $X_2 = s^x X_1$. The Lagrange multipliers $\lambda_1 = -p^y q g(V_1) + (\xi / b s^y \varphi)$ and $\lambda_2 = -p^y q g(V_2)$ are then used to rewrite the first order condition (7) as $\dfrac{\partial \Pi_1}{\partial X_1} - p^y q g(V_1) s^x - \mu_1 \dfrac{\partial V_2}{\partial X_1} + \beta^{-1} p^y q g(V_1) - \dfrac{s^x \xi}{b s^y \varphi} = 0$,

$\dfrac{\partial \Pi_2}{\partial X_2} - \dfrac{p^y q g(V_2)}{s^x} - \mu_2 \dfrac{\partial V_1}{\partial X_2} + \beta^{-1} p^y q g(V_1) + \beta^{-1} \dfrac{\xi}{b s^y \varphi} = 0$, where

$\mu_1 = (\partial V_2 / \partial V_1) \partial \Pi_1 / \partial V_1 - \beta^{-1} \partial \Pi_2 / \partial V_2$, and $\mu_2 = -\beta^{-1} \partial \Pi_1 / \partial V_1 + (\partial V_1 / \partial V_2) \partial \Pi_2 / \partial V_2$

Therefore, the optimal pulse slaughtering cyclical equilibrium is given by the following three equations

$$\frac{\partial \Pi_1}{\partial X_1} - p^y q g(V_1) s^x - \mu_1 \frac{\partial V_2}{\partial X_1} + \beta^{-1} p^y q g(V_1) = \frac{\beta^{-1}}{s^x} \left(\frac{\partial \Pi_2}{\partial X_2} - \frac{p^y q g(V_2)}{s^x} - \mu_2 \frac{\partial V_1}{\partial X_2} + \beta^{-1} p^y q g(V_1) \right)$$

, $V_1 = V_2 + f(V_2) - g(V_2)(1+b)s^x X_1$ and $V_2 = V_1 + f(V_1) - g(V_1)(1+b)X_1$. These equations characterize the number of animals X_1 and the vegetation quantities V_1 and V_2, given that $h_1^y = 1$, $h_2^y = \left[\psi b s^y + s^y - (1/s^x) \right] / \psi b s^y$ and $X_2 = s^x X_1$.

6. Numerical illustration

6.1 Data and functional forms

To shed further light on the above analysis, the model is illustrated numerically. As already indicated, the specification of the animal consumption function follows Noy-Meir (1975), $g(V_t) = kV_t / (V_t + m)$, where $k > 0$ is the maximum vegetation biomass intake per animal and $m > 0$ determines the shape of the consumption pattern. Natural vegetation growth is described by the standard logistic function $f(V_t) = rV_t(1 - V_t/Q)$, with $r > 0$ as the maximum specific vegetation growth rate (vegetation productivity) and $Q > 0$ as the carrying capacity. As mentioned (section two), the vegetation quantity may therefore in principle approach zero. However, as also mentioned, overgrazing is

not a serious problem in our species-rich alpine environments. The animal cost function is specified as linear, $C(X_t) = cX_t$, with $c > 0$ as the fixed marginal cost (see also above).

The baseline parameter values, in which sheep biological data and economic values are related to Norwegian conditions, are shown in Table A1 (Appendix 2). The sheep biological data are based on a large set of observations, while the vegetation parameter values, at least to some extent, are based on qualified guesswork. The size of the farm is scaled by the vegetation carrying capacity Q. With $Q = 500$ (ton of vegetation biomass), we find $V^{msy} = Q/2 = 250$. Accordingly, for the baseline parameter values (Table A1), the number of animals (and winter population) in a possible equilibrium $f(V) = g(V)(1+b)X$, or $rV(1-V/Q) = [kV/(V+m)](1+b)X$ compatible with this V^{msy} value, is $X = r(Q+2m)/4k(1+b) = 108$ (ewes). This corresponds to a farm somewhat above the Norwegian average size. For this vegetation quantity, the lamb weight is $w^y = qg(V^{msy}) = 21.8$ (kg/animal), and the per lamb meat value $p^y qg(V^{msy}) = 50 \cdot 21.8 = 1,091$ (NOK/animal). The ewe slaughter value is fixed as $p^x w^x = 35 \cdot 30 = 1,050$ (NOK/animal) (Table A1). A higher slaughter value of the lambs than that of the ewes is the typical situation according to market data. For our baseline parameter values, the marginal maintenance cost ($c = 650$ NOK/animal) exceeds the per animal wool income ($z \cdot \theta = 35 \cdot 5 = 175$ NOK/animal).

We find the optimal management policy for the baseline parameter values. Effects of changes in the discount rent and costs are also studied. We start with the stationary solutions.

6.2 Results stationary solutions

Table 1 demonstrates the three stationary solutions; that is, the two steady states and the optimal pulse harvesting. In the baseline case, just as in reality, the value per animal is higher for the lambs than the ewes, and hence the steady state profit is higher when only lambs are slaughtered (Proposition 1). However, the difference is quite modest, just about 4-5 % (106.26 vs.101.95). The harvesting rates are 0.928 and 0.680, respectively, and the stocking rate is highest and the vegetation quantity lowest when only ewes are

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slaughtered. With only lamb slaughtering, we find the vegetation quantity to be above that of V^{msy} ($V = 268 > V^{msy} = 250$). On the other hand, with only ewes slaughtering, the vegetation quantity is substantial lower, with $V = 221$.

Table 1 about here

The last two columns in Table 1 demonstrate that pulse harvesting with only lamb slaughtering represents the most beneficial stationary harvesting strategy. The two-year discounted profit exceeds the steady state with only lambs slaughtered (Proposition 3), but note that the economic benefit of a stationary cycle policy is low. The lamb harvesting rates in the two consecutive years are 1.000 and 0.8526, while the number of animals is 107 and 101, respectively. Note also that stationary cycles in period one sustains a higher level of vegetation as well as higher lamb weight than in the steady state solution. That is, the cyclical solution can be understood as optimal fallows for increasing lamb weight and vegetation level in periods with high slaughtering rates. Pulse fishing plays the same role in fisheries models (see Da Rocha et al. 2011).

In our farm model, there are no stock dependent harvesting costs. The costs are maintenance costs related to the number of animals kept during the winter indoors season. Therefore, we find that the advantage of cyclical solutions do not rely on the size of these costs. See Table 2. As Proposition 3 states, even for high maintenance costs, pulse cycle periodic harvesting policy is always better than the steady state solution with a constant number of lambs slaughtered every year. However, again we find that the economic gain is small. As expected, we also find that higher costs mean that it is beneficial to reduce the stocking rate and reduce the grazing pressure.

Table 2 about here

Next, in Table 3, we show how the two-year discounted profit changes with the size of the discount rate. Regardless of the size of the discount rate, pulse cycle periodic harvesting policy is always better than the steady states. Moreover, we also find that the number of animals increases and the vegetation quantity decreases in the steady state solution with only lamb slaughtering with a more myopic farm policy and a higher discount rent. This result highlights the different role that each state variable plays in an

animal vegetation system. The animals are predators that harvest vegetation, the prey. That is, in such a system, vegetation is the natural resource, while the animals represent the capital, and, at a higher discount rent, it is beneficial for the farmer to invest in the animal capital and not 'in the bank'. The same occurs in the steady state solution with only ewes slaughtering as well as with pulse slaughtering.

Table 3 about here

6.3 Dynamic transition

Above it was shown that a pulse cyclical periodic solution was beneficial compared to the ewes-only steady state harvesting scheme with a low vegetation quantity when the discount rate was high and $\delta = 0.10$. However, we may find that selecting the low vegetation steady state may represent the optimal policy by depleting the initial level of vegetation. To show this, we compute the transitional dynamics using dynamic programming. To find the optimal control rule associated with any possible state, we use Proposition 2 to write the controls as functions of the state variables, and we solve the Bellman equation $\upsilon(V_t, X_t) = \max_{V_{t+1}, X_{t+1}} \Pi(V_{t+1}, X_{t+1} \mid V_t, X_t) + \beta \upsilon(V_{t+1}, X_{t+1})$.

Given the optimal policies, transitional dynamics are computed for three different initial conditions; i) the 'high' vegetation steady state (left hand side panels Figures 2 and 3), ii) the 'low' vegetation steady state (middle panels) , and finally iii) the first year of the optimal pulse cycle (right hand side panels). In each panel, where Figure 4 indicates the stock values and Figure 5 the harvesting rates, the dynamic transitions are shown for three different values of the discount rent; solid line for $\delta = 0.03$ (baseline value), solid line with a cross mark for $\delta = 0.07$ and dashed line for $\delta = 0.10$. Finally, in Figure 4's bottom panels, demonstrating the optimal vegetation paths, we have also included a solid line indicating the vegetation level for which the value of slaughtering lambs equalizes the value of ewes; that is, $p^y qg(V) = p^x w^x$ and hence $V \approx 234$. Therefore, slaughtering only lambs (ewes) is optimal if the vegetation quantity is above (below) this level.

Figure 2 about here

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For the baseline value of the discount rate and $\delta = 0.03$, the stationary solution selected is the cyclical pulse solution irrespective of the initial conditions. In Figure 4, the lower panels, the three transitional dynamics lead to a vegetation level above that of $V \approx 234$. Therefore, irrespective of the initial conditions, it is optimal to reach the periodic cycle stationary solution, and depleting the initial level of vegetation is not optimal.

On the other hand, when the discount rate is high and $\delta = 0.10$, the optimal stationary solution selected is always the steady state solution with slaughtering of only ewes and hence a low vegetation level. Therefore, in Figure 4 (bottom panels), the three transitional dynamics lead to a vegetation level of about 190, well below the solid line indicating the vegetation level of similar slaughter values of lambs and ewes ($V \approx 234$). The steady state number of animals becomes $X = 120$ and the slaughter rate $h^x = 0.68$. The long run solution selected is the steady state with the lowest discounted profit (again, see Table 3); the reason is that depleting the initial high vegetation stock more than outweighs future steady state losses. This result is similar to the effect of the discount rate in a single stock model. For a high interest rate, depletion along the transitional path compensates for the future stationary losses.

Figure 3 about here

Finally, when the discount rate is $\delta = 0.07$, we find that the optimal stationary solution depends on the initial conditions. In Figure 2, lower panels, two of the three transitional dynamics paths go to a vegetation level above the $V \approx 234$ line, and one goes below. Therefore, if we start with a low level of vegetation as indicated by the stationary solution of only ewes slaughtering (middle panels), depletion of the vegetation quantity is optimal. In the other two cases, however, the stationary solution selected is the cyclical pulse solution with only lamb slaughtering and h^y oscillating between 1.00 and 0.84. See Figure 3. Table 4 summarizes the above results.

Table 4 about here

7. Concluding Remarks

This paper has analyzed the economics of sheep farming in a two-stage model of lambs and adult females (ewes). The analysis is at the farm level in a Nordic context with a crucial distinction between the outdoor grazing season and the winter indoor feeding season. A Noy-Meir (1975) type model describes the animal – vegetation interaction during grazing season where the vegetation growth is density dependent, while the animal growth is linear and hence does not depend on the vegetation quantity.. The vegetation quantity and food intake during the outdoors grazing season influences the weight gain and slaughter weight of the lambs. The farmer is assumed to be rational and well informed, and aims to find the animal slaughter composition maximizing the present- value profit comprising meat (flow) value and wool (stock) value.

The paper provides three propositions about the optimal slaughter decision. In a possible steady state, Proposition 1 says that the harvesting decision is shaped by economic factors alone. With a 'high' vegetation level and more valuable lambs than adults, slaughtering only lambs is optimal. The slaughtering should take place at the highest level compatible with population equilibrium determined by only biological factors (survival rates and fertility). In a possible steady state with a 'low' vegetation level, slaughtering only ewes is optimal, and this should again take place at the highest level compatible with population equilibrium, also determined by biological factors alone. We are also able to say something about the dynamics and transitional paths, conveyed by Proposition 2, indicating that, along the transitional paths, the optimal harvest policy no longer depends only on biological factors. The harvest policy is also contingent upon economic factors working through the number of animals.

The last result of our analysis is stated as Proposition 3, and says that pulse (cyclical) slaughtering with only lamb slaughtering may be optimal. Our mechanism for cyclical harvesting is different from what is found in most of the existing literature. See, e.g., Wirl (1995), and Tahvonen (2009) who demonstrates pulse harvesting under imperfect harvesting selectivity. Pulse fishing has some advantages in live product fisheries (Da Rocha, Gutierrez and Antelo 2012a, b), and it has been applied in timber management under spatial rotation. Nevertheless, sheep farmers do not consider periodic slaughtering as a feasible management tool. In our numerical analysis, we also find that the economic advantage of periodic slaughtering is small compared to stationary slaughtering.

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Appendix 1
Proof of Proposition 2

Assume that it exists an optimal solution where X_{t+1} / X_t is greater than s^x. Given that h_t^y must be less than one when X_{t+1} / X_t is greater than s^x, the only possible option is lamb slaughtering; that is, $h_t^x = 0$ and $h_t^y = 1 - [(X_{t+1} / X_t) - s^x] / b\Psi s^y$. We prove by contradiction that this is the optimal policy. Assume that exists an optimal solution where X_{t+1} / X_t is greater than s^x and h_t^x is strictly positive, and let

$\Pi(X_t, V_t, h_t^y, h_t^x) = p^y qg(V_t) bX_t s^y (\psi h_t^y + 1 - \psi) + p^x w^x X_t s^x h_t^x + z\theta X_t - C(X_t)$ be the optimal current profit associated to this solution. Now consider a new sequence of controls with

$\hat{h}_t^x = 0$, and $\hat{h}_t^y = h_t^y + \dfrac{s^x}{bs^y \psi} h_t^x$ that sustain the same sequence of number of

animals X_t and therefore also the same vegetation quantity V_t (cf. Eq. 1). We now find

that $\Pi(X_t, V_t, \hat{h}_t^y, \hat{h}_t^x) - \Pi(X_t, V_t, h_t^y, h_t^x) = (p^y qg(V_t) - p^x w^x) X_t s^x h_t^x$ is positive when

$p^y qg(V_t) > p^x w^x$, and we hence get a contradiction. If the optimal solution is such that X_{t+1} / X_t is lower than s^x and h_t^y is lower than 1, we can increase h_t^y by reducing h_t^x in a

quantity equal to $\dfrac{bs^{y}\psi}{s^{x}}h_{t}^{y}$ for keeping the same number of animals. Therefore, consider a

new sequence of controls where $\hat{h}_{t}^{x}=1-\dfrac{(X_{t+1}/X_{t})}{s^{x}}$ and $\hat{h}_{t}^{y}=1$. We find a contradiction

also because $\Pi(X_{t},V_{t},\hat{h}_{t}^{y},\hat{h}_{t}^{x})-\Pi(X_{t},V_{t},h_{t}^{y},h_{t}^{x})=(p^{y}qg(V_{t})-p^{x}w^{x})X_{t}bs^{y}\psi(1-h_{t}^{y})>0$.

Proof of Proposition 3
We show that, given a steady state sequence, we can always construct a stationary cycle that improves the net present profits. Assume that the optimal solution is a steady
state $X_{t+1}=X_{t}=X$ and $V_{t+1}=V_{t}=V$ with $h^{x}=0$ and $h^{y}=1-(1-s^{x})/\psi bs^{y}$. First, we build a feasible stationary cycle, by modifying the number of animals in the second
year, $X_{2}=(1-\varepsilon)X$ and keeping constant the number of animals in the first
year, $X_{1}=X$. Changes in the number of animal imply changes in: i) the slaughtering rates, and ii) the sequence of vegetation, for guarantying feasibility. Formally, a
stationary cycle with $X_{2}=(1-\varepsilon)X$ implies (from proposition 1) $\hat{h}_{1}^{y}=h^{y}+\dfrac{\varepsilon}{\psi bs^{y}}$,

$\hat{h}_{2}^{y}=h^{y}-\dfrac{\varepsilon}{\psi bs^{y}(1-\varepsilon)}$. Second, the new vegetation sequence, associated with

$X_{2}=(1-\varepsilon)X$, satisfies the feasibility conditions
$\hat{V}_{2}=\hat{V}_{1}+f(\hat{V}_{1})-(1+b)g(\hat{V}_{1})X$ and $\hat{V}_{1}=\hat{V}_{2}+f(\hat{V}_{2})-(1+b)g(\hat{V}_{1})(1-\varepsilon)X$.

Now, we can compute the difference between profits associated with the stationary cycle and the steady state sequences in the first year of the stationary cycle
$$\Delta\Pi_{1}=\Pi(X_{1},\hat{V}_{1},\hat{h}_{1}^{y})-\Pi(X,V,h^{y})=p^{y}q\left\{bs^{y}\left[g(\hat{V}_{1})-g(V)\right](\psi h^{y}+1-\psi)+g(\hat{V}_{t})\varepsilon\right\}X$$
and in the second year
$$\Delta\Pi_{2}=\Pi(X_{2},\hat{V}_{2},\hat{h}_{2}^{y})-\Pi(X,V,h^{y})=p^{y}q\left\{bs^{y}\left[(1-\varepsilon)g(\hat{V}_{2})-g(V)\right](\psi h^{y}+1-\psi)-g(\hat{V}_{2})\varepsilon\right\}X$$
$-z\theta\varepsilon X+c\varepsilon X$. We can compute the difference in the net present profits, $\Delta\Pi_{1}+\beta\Delta\Pi_{2}$ as
$$\Delta\Pi_{1}+\beta\Delta\Pi_{2}=p^{y}qbs^{y}(\psi h^{y}+1-\psi)\left[g(\hat{V}_{1})+\beta(1-\varepsilon)g(\hat{V}_{2})-(1+\beta)g(V)\right]X$$
$+\left[g(\hat{V}_{1})-\beta g(\hat{V}_{2})-\beta z\theta\right]\varepsilon X+\beta c\varepsilon X$. Therefore
$$\frac{d(\Delta\Pi_{1}+\beta\Delta\Pi_{2})}{d\varepsilon}=p^{y}qbs^{y}(\psi h^{y}+1-\psi)\left[\frac{dg(\hat{V}_{1})}{d\varepsilon}+\beta(1-\varepsilon)\frac{dg(\hat{V}_{2})}{d\varepsilon}-\beta g(\hat{V}_{2})\right]X$$
$$+\left[g(\hat{V}_{1})-\beta g(\hat{V}_{2})+\frac{dg(\hat{V}_{1})}{d\varepsilon}\varepsilon-\beta\frac{dg(\hat{V}_{2})}{d\varepsilon}\varepsilon\right]X-\beta[z\theta-c]X$$
where
$$\frac{dg(\hat{V}_{1})}{d\varepsilon}=-g'(\hat{V}_{1})\frac{\partial\varphi(\hat{V}_{1},\hat{V}_{2},\varepsilon)/\partial\varepsilon}{\partial\varphi(\hat{V}_{1},\hat{V}_{2},\varepsilon)/\partial\hat{V}_{1}}=g'(\hat{V}_{1})\frac{(1+b)g(\hat{V}_{2})X}{\left[f'(\hat{V}_{1})-(1+b)g'(\hat{V}_{1})\right]}$$
and

$$\frac{dg(\widehat{V}_2)}{d\varepsilon} = -g'(\widehat{V}_2)\frac{\partial\varphi(\widehat{V}_1,\widehat{V}_2,\varepsilon)/\partial\varepsilon}{\partial\varphi(\widehat{V}_1,\widehat{V}_2,\varepsilon)/\partial\widehat{V}_2} = g'(\widehat{V}_2)\frac{(1+b)g(\widehat{V}_2)X}{\left[f'(\widehat{V}_2) - (1+b)g'(\widehat{V}_1)(1-\varepsilon)\right]}$$

are calculated using the relationship between vegetation in the odd and even periods given by $\varphi(\widehat{V}_1,\widehat{V}_2,\varepsilon) = f(\widehat{V}_1) + f(\widehat{V}_2) - (1+b)[g(\widehat{V}_1) + g(\widehat{V}_2)(1-\varepsilon)]X$. Therefore, when ε goes to zero $\widehat{V}_1 = \widehat{V}_2 = V$ and

$$\lim_{\varepsilon\to 0}\frac{d(\Delta\Pi_1 + \beta\Delta\Pi_2)}{d\varepsilon} = p^y qbs^y(\psi h^y + 1 - \psi)\left[(1+\beta)\frac{dg(V)}{d\varepsilon}\bigg|_{\varepsilon\to 0} - \beta g(V)\right]X + (1-\beta)g(V)X + \beta\left[c - z\theta\right]X$$

Then, given that $(1-\beta)g(V)X + \beta[c - z\theta]X$ is positive, we need to show that

$$(1+\beta)\frac{dg(V)}{g\varepsilon}\bigg|_{\varepsilon\to\infty} - \beta g(V) > 0. \quad \text{Then, given that}$$

$$(1+\beta)\frac{dg(V)}{g\varepsilon}\bigg|_{\varepsilon\to\infty} - \beta g(V) = \frac{-g(V)(1+b)X}{f'(V) - (1+b)g'(V)X}\left[g'(V) + \beta f'(V)\frac{g(V)}{f(V)}\right]$$

if $g'(V) > \beta f'(V)\dfrac{g(V)}{f(V)}$ steady states are not optimal.

Appendix 2
Data and parameter values

Aunsmo et al. (1998) and Nersten et al. (2003) provide economic data. Prices and costs are in 2003 values. The sheep biological baseline parameter values are based on Mysterud et al. (2002) and Aunsmo et al. (1998). As a background for the vegetation growth values, there are some studies indicating the amount of fodder production. However, alpine pastures are heterogeneous, and estimations of fodder production from two alpine ranges in Norway (Setesdalsheiene and Hardangervidda) show large variations. Vegetation types with a limited biomass production dominate. There are also meadows that produce a large amount of fodder of very high quality. However, the meadows cover only a small proportion of these areas (Austrheim et al. 2008a). The vegetation consumption values build on detailed animal food intake data, but are also calibrated based on weight and price data. This is also true for the vegetation growth values. For details and calculations see Skonhoft et al. (2010). Based on these calculations, the vegetation saturation parameter is fixed as $k = 0.50$ (ton of vegetation biomass/animal), the intrinsic vegetation productivity parameter value is assumed to be $r = 0.5$ while the animal consumption shape parameter value set to $m = 300$ (ton of vegetation biomass). The farm size is scaled through the vegetation carrying capacity given as $Q = 500$ (ton) (see numerical section). Based on the ewe weight of $w^x = 30$ (kg/animal) and the fact that the ewe weight should be above that of the lamb weight for all values of the vegetation quantity, the value of the biomass translation parameter is assumed to be $q = 96$ (kg meat/ton vegetation biomass). Table A1 gives the baseline parameter values used in the numerical illustrations.
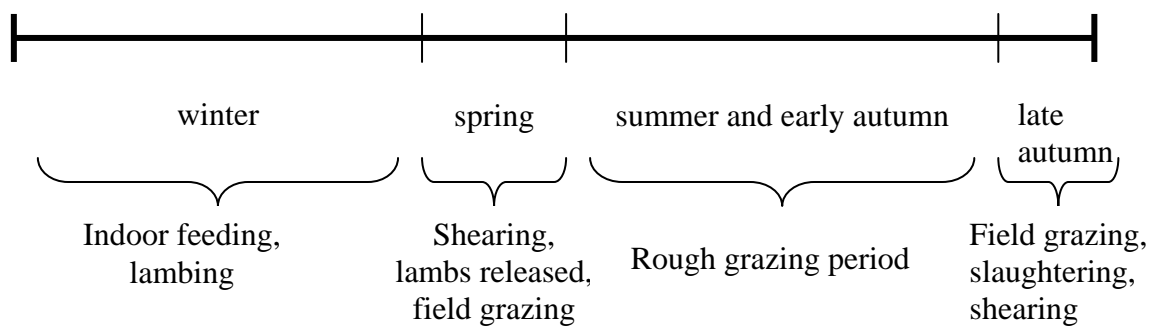
Table A1 about here

Tables and Figures

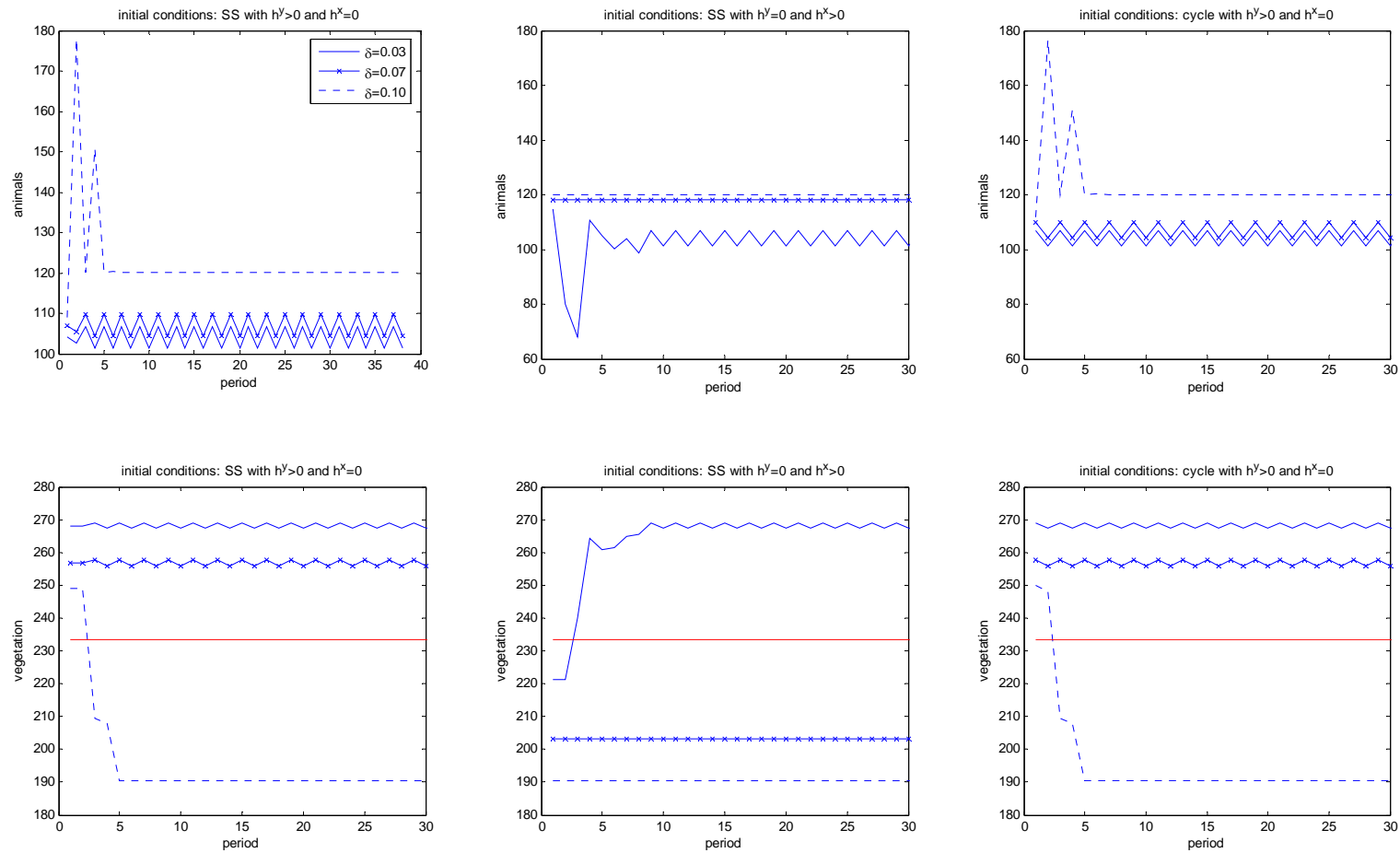**Figure 1: Seasonal subdivision in the Nordic sheep farming system.**

Figure 2. Optimal animal and vegetation paths for three different initial conditions, and three different discount rates. Steady state (SS) with $h^y > 0$ (left), steady state (SS) with $h^x > 0$ (center) and pulse (right). Initial conditions as in the stationary solutions for baseline discount rate $\delta = 0.03$ (Table 1)
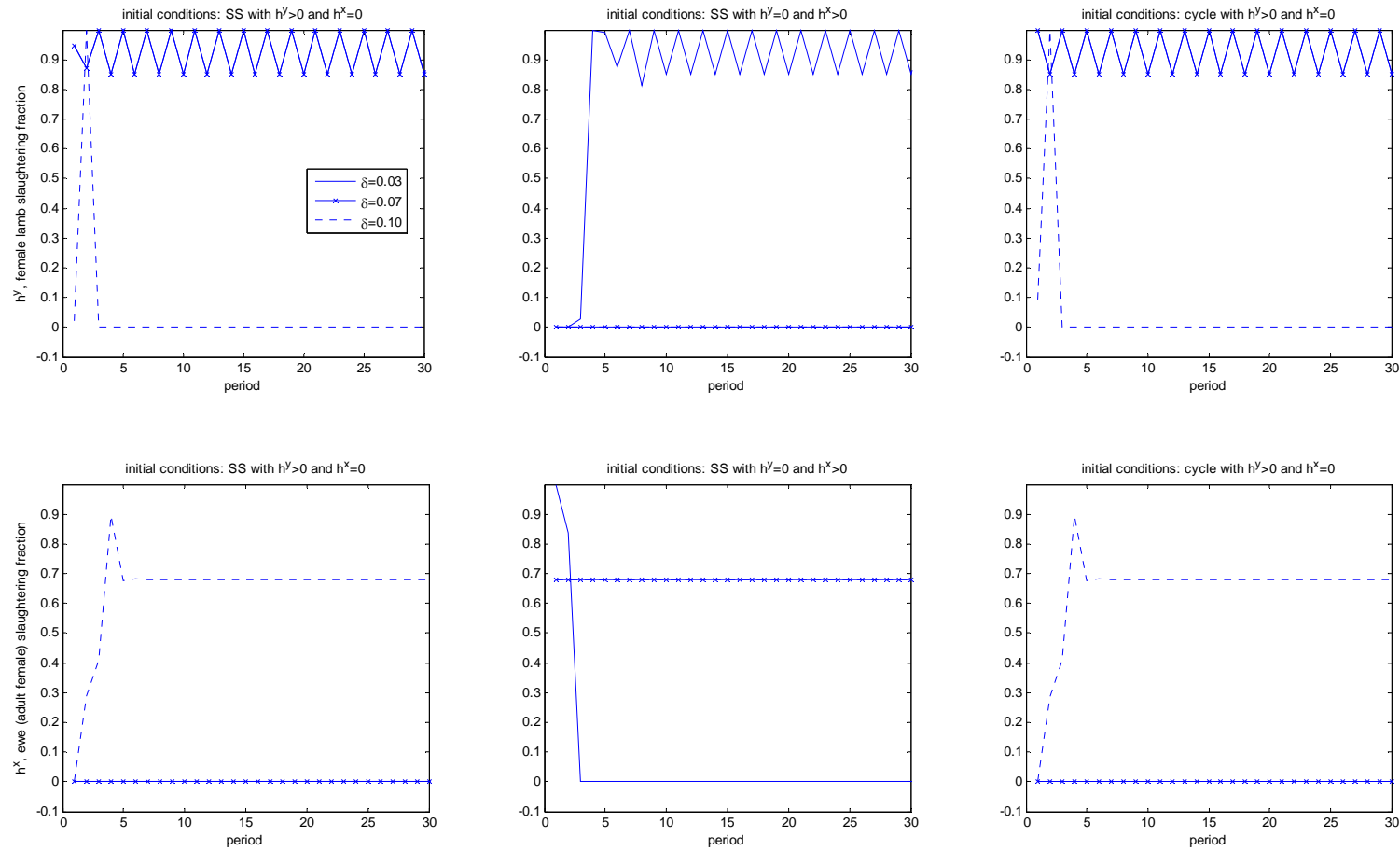
Figure 3. Optimal slaughtering paths for three different initial conditions, and three different discount rates. Steady state (SS) with $h^y > 0$ (left), steady state (SS) with $h^x > 0$ (center) and pulse (right). Initial conditions as in the stationary solutions for baseline discount rate $\delta = 0.03$ (Table 1)

Table 1: Stationary solutions. Baseline parameter values

	Steady state. Only lamb slaughtering	Steady state. Only ewe slaughtering	Pulse harvesting year 1	Pulse harvesting year 2
Number of animals X	104	115	107	101
Vegetation quantity (ton) V	268	221	269	267
Lamb weight (kg) w^y	22.65	20.37	22.69	22.61
Slaughtering fraction lambs h^y	0.928	0	1.000	0.853
Slaughtering fraction ewes h^x	0	0.680	0	0
Animal growth rate	0	0	0.950	1.053
Current profit Π (1,000 NOK)	106.26	101.95	115.32	97.23
2-period discounted profit (1,000 NOK)	209.43	200.94	-	209.71

Table 2: Optimal stationary solutions. Effects of changing costs. All other parameter values as baseline

	$c = 650/2$	Baseline $c = 650$	$c = 650 \cdot 2$
	Steady state with only lambs slaughtering		
Number of Animals X	110	104	83
Vegetation quantity (ton) V	246	268	334
2-period discounted profit (1,000 NOK)	278.49	209.43	85.61
	Steady state with only ewes slaughtering		
Number of Animals X	119	115	91
Vegetation quantity (ton) V	196	221	311
2-period discounted profit (1,000 NOK)	276.33	200.94	63.67
	Pulse		
Number of Animals X (year 1)	112	107	85
Vegetation (ton) V (year 1)	247	269	335
Number of Animals X (year 2)	107	101	81
Vegetation (ton) V (year 2)	245	267	333
2-period discounted profit (1,000 NOK)	278.80	209.71	85.88

Table 3: Optimal stationary solutions. Effects of changing discount rate. All other parameter values as baseline

	Baseline $\delta = 0.03$	$\delta = 0.07$	$\delta = 0.10$
Steady state with only lamb slaughtering			
Number of Animals X	104	107	109
Vegetation quantity (ton) V	268	257	249
2-period discounted profit (1,000 NOK)	209.43	204.13	199.88
Steady state with ewes			
Number of Animals X	115	118	120
Vegetation quantity (ton) V	221	203	190
2-period discounted profit (1,000 NOK)	200.94	194.84	189.52
Pulse with only lamb slaughtering			
Number of Animals X (year 1)	107	110	112
Vegetation (ton) V (year 1)	269	258	250
Number of Animals X (year 2)	101	104	106
Vegetation (ton) V (year 2)	267	256	248
2-period discounted profit (1,000 NOK)	209.71	204.75	200.73

Table 4: Optimal stationary solution selected in the long run. Number of animals and vegetation quantity

Discount rent	$\delta = 0.03$	$\delta = 0.07$	$\delta = 0.10$
Scenario1: Initial condition Steady state with lamb slaughtering			
Animal period 1	107	110	120
Animal period 2	101	104	120
Vegetation period 1	269	258	190
Vegetation period 2	267	256	190
Scenario 2: Initial conditions Steady State with ewe slaughtering			
Animal period 1	107	118	120
Animal period 2	101	118	120
Vegetation period 1	269	203	190
Vegetation period 2	267	203	190
Scenario 3: Initial conditions Pulse cyclical equilibrium			
Animal period 1	107	110	120
Animal period 2	101	104	120
Vegetation period 1	269	258	190
Vegetation period 2	267	256	190

Table A1: Baseline ecological and economic parameter values.

Parameter	Parameter description	Value
s^y	-Natural survival fraction lambs	0.91
s^x	-Natural survival fraction ewes	0.95
b	-Fertility rate	1.53 (lamb/ewe)
ψ	- Proportion female lambs	0.50
w^x	-Adult (ewe) slaughter weight	30 (kg/animal)
θ	-Wool output coefficient	5 (kg/animal)
z	-Wool price	35 (NOK/kg)
q	-Biomass translation parameter	96 (kg meat/ ton vegetation biomass)
r	-Intrinsic vegetation growth rate (pasture productivity)	0.50
Q	-Vegetation carrying capacity	500 (ton of vegetation biomass)
k	-Vegetation saturation parameter	0.50 (ton of vegetation biomass/ animal)
m	-Shape animal consumption parameter	300 (ton of vegetation biomass)
p^x	-Adult (ewe) slaughter price	35 (NOK/kg)
p^y	-Lamb slaughter price	50 (NOK/kg)
c	-Marginal cost	650 (NOK/animal)
δ	-Discount rent	0.03

Table note: Exchange rate: 1 Euro = 7.90 NOK (Summer 2011).